OKA GEOMETRY AND DEFORMATIONS OF TORUS–FIBRED COMPLEX SIX-SPHERES

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Abstract. Starting from Alpöge's torus fibration with exactly three special fibres on a complex six-sphere, we construct a locally miniversal family of compact complex threefolds. Every member is Oka, and the projection over each relatively compact parameter disc is an Oka map. Deleting or simultaneously blowing up finitely many distinct points preserves the Oka property. We also determine the tangent cohomology, the smooth Kuranishi germ, the automorphism group, and the exact biholomorphism relation $X_a \simeq X_{a'}$ if and only if $a' - a \in 2\mathbf{Z}$. The Oka arguments use explicit cyclic covers and Kusakabe's localization principle.

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1. Introduction

The Oka principle begins with a flexibility question: when can a continuous map from a Stein space be deformed to a holomorphic map? A complex manifold is *Oka* if these deformations can be made with approximation, interpolation, and parameters [20, 23]. Gromov proved, using holomorphic sprays, that elliptic manifolds are Oka [1]. For a holomorphic submersion, an *Oka map* is the relative version: liftings over a fixed map to the base enjoy the same flexibility, and the submersion is a Serre fibration [24]. Kusakabe's analytic-Zariski localization theorem allows flexibility to be established on large holomorphic opens and then assembled across their analytic complements [21]. His Oka-map criterion extends this strategy to holomorphic submersions [24]. These tools are well suited to a fibration whose general fibre is homogeneous while its special fibres are singular or multiple.

Alpöge constructed a torus-fibred complex threefold diffeomorphic to S^6 [2]. Its general fibre is a complex two-torus, and the fibration has exactly three special values on $\mathbf{P}^1$: two multiple fibres with bielliptic reductions and one irreducible nonnormal cusp fibre, whose normalization is a degree-six del Pezzo surface. Any ordered triple of distinct values on $\mathbf{P}^1$ can be moved to $0, 1, \infty$ by an automorphism; these are the coordinates used below. The orders 3 and 4 and the nature of the cusp remain essential to the construction and to the exact-period calculation. We use Alpöge's construction and smooth-type theorem, including its complex structure on S^6; Appendix D records the particular source ingredients used here and their locations.

The smooth fibres and $\mathbf{P}^1$ are Oka, which makes the Oka property of the total space a natural question. The smooth-fibre base $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ is hyperbolic, and the three special fillings are decisive for the compact threefold. Our proof uses the cyclic covers of these fillings and Kusakabe's localization principle.

An additive period direction produces a half-plane $\mathcal{A}$. Over each relatively compact parameter disc $\Delta \Subset \mathcal{A}$ the fixed filling data give a holomorphic family $\pi_\Delta : \mathfrak{X}_\Delta \to \Delta$ of compact threefolds X_a. Its marked matrix is

$$\Omega_a(z) = \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta_0(z) + a & \mu(z) & 0 & 1 \end{pmatrix}, \qquad \mathcal{A} = \left\{ a : \Im a < -\sup_z \left(\Im \beta_0(z) - \frac{6(\Im \mu(z))^2}{\Im \tau(z)} \right) \right\}.$$

The construction fixes the monodromy, finite logarithmic twists and untranslated cusp gluing as a varies. Its proper submersion and smooth-type comparison are proved in Section 2. Write $f_a : X_a \to \mathbf{P}^1$ for the torus fibration and $S_{1,a}, S_{2,a}$ for the two finite reductions. Thus

$$f_a^*(0) = 3S_{1,a}, \qquad f_a^*(1) = 4S_{2,a}, \qquad f_a^{-1}(\infty) = W_a. \tag{1.1}$$

Theorem 1.1. *For every disc $\Delta \Subset \mathcal{A}$, the projection $\pi_\Delta : \mathfrak{X}_\Delta \to \Delta$ is an Oka map. For every $a \in \mathcal{A}$, the manifold X_a and both $X_a \setminus S_{1,a}$ and $X_a \setminus S_{2,a}$ are Oka.*

The two complementary finite-fibre deletions let us approach every special filling in a model where the other one is absent. After finite base change, a cyclic period-lattice cover carries analytic-Zariski translates of a two-row conic band. Its regular part is a suspension over an elliptic surface with Danielewski charts; the singular conics are repaired by a small resolution. A normalized double cover and four corrected incidence charts recover the exceptional curves that the regular suspension misses. Complete vector fields prove flexibility on those charts, and Kusakabe localization assembles the band, each finite-fibre complement, and finally X_a.

The family assertion requires more than fibrewise Oka. The chosen period-lattice kernel has periods independent of a, so its covering family over a disc is a product with one fixed band model. A spray on the fixed factor moves lifted maps vertically while holding the parameter fixed. Descent and relative localization give the Oka property of π_Δ. The detailed cyclic-cover construction and its relative spray argument occupy Section 3.

Removing or blowing up points tests the flexibility of this particular geometry. The complete lift of a finite centre has a precise form: in each two-row band it is a finite union of full deck orbits in the common core together with finitely many toroidal points. The distinction is essential: retaining only one representative of a core orbit would destroy covering descent.

Theorem 1.2. *For every $a \in \mathcal{A}$ and every finite set $P \subset X_a$ of distinct points, both $X_a \setminus P$ and the simultaneous blow-up $\mathrm{Bl}_P X_a$ are Oka.*

The proof uses proper monomial characters on the algebraic carrier curves of the lifted core orbits. Character-adapted logarithmic covers turn their complete inverse images into tame discrete sets. At the residual axes, complete replicas move finite centres into principal charts while fixing the wrong-ruling lines. For blow-ups this must be done on the *same* centre in every overlapping chart: flat base change, finite Galois descent and analytic-Zariski localization then yield the band blow-up and, after covering descent, the compact blow-up. The order-4 incidence coordinate remains a section of a possibly nontrivial line bundle; none of the estimates assumes a global trivialization. Full proofs are in Sections 4 and 5.

For a single point p, the Oka manifold $\mathrm{Bl}_p X_a$ is smoothly diffeomorphic to $\mathbf{P}^3$. Its complex structure differs from the standard projective one: the first Chern class of its tangent bundle has divisibility 2, whereas that of the standard $\mathbf{P}^3$ has divisibility 4. The topology and Chern-class calculation are spelled out in Corollary 5.19.

The band geometry also supplies curve complements. If any chosen subcollection of the canonical exceptional curves is deleted from a two-row band, the result remains Oka, even together with the mixed finite/deck point deletions used above. The complement of the complete residual-axis curve configuration is Oka as well (Theorems 4.56 and 4.57). These are band statements. For an arbitrary smooth compact curve $C \subset X_a$, the published blow-up theorem of Forstnerič–Lárusson gives $\mathrm{Bl}_C X_a$ the Oka–1 property (Corollary 5.20); the local $(-1,-1)$ conifold model has an algebraically elliptic blow-up (Proposition 5.21).

To understand whether this period direction captures all nearby complex structures, we compute ordinary tangent cohomology rather than infer completeness from the family dimension. The order-3 and order-4 stabilizers are recorded on a root-stack base; at the cusp, the branchwise normal quotient of the degree-six del Pezzo normalization supplies the remaining contribution. The calculation gives $(h^0, h^1, h^2, h^3)(X_a, T_{X_a}) = (1, 1, 1, 0)$. In particular the obstruction space is nonzero. Differentiating the actual gluing maps produces a Kodaira–Spencer class that generates H^1. The resulting classifying map forces the one-dimensional Kuranishi germ to be smooth and reduced.

Theorem 1.3. *For every $a \in \mathcal{A}$, the Kodaira–Spencer map $T_a\mathcal{A} \to H^1(X_a, T_{X_a})$ is an isomorphism. The Kuranishi germ of X_a is a smooth reduced line, and the family is locally miniversal. Moreover $H^0(X_a, T_{X_a}) = \mathbf{C}V$ and $\mathrm{Aut}^0(X_a) \simeq \mathbf{C}^*$.*

Consequently, every sufficiently small proper holomorphic deformation family of a marked X_a is locally a pullback of the period family. Oka maps remain Oka under pullback, so such a family has an Oka projection near that fibre (Corollary 7.16).

The parameter also has a global meaning. The complete bi-anticanonical pencil recovers f_a from X_a itself. The topology of S^6 gives a complementary rigidity statement: every holomorphic map $X_a \to \mathbf{P}^1$ factors through f_a. Indeed, $H^2(X_a, \mathbf{Z}) = 0$ forces the restriction of such a map to each smooth Kähler torus fibre to be constant, and a proper-image argument extends the factor across the three special values. The unequal multiplicities and the nonnormal cusp then force any biholomorphism between two members to preserve the base coordinate. The common monodromy centralizer allows only an integral period shift; the finite logarithmic twists exclude the odd shifts, and explicit maps realize every even shift.

Theorem 1.4. *For all $a, a' \in \mathcal{A}$,*

$$X_a \simeq X_{a'} \iff (X_a, f_a) \simeq_{\mathbf{P}^1} (X_{a'}, f_{a'}) \iff a' - a \in 2\mathbf{Z}.$$

In particular $\mathrm{Aut}(X_a) = \mathbf{C}^*$.

Thus a disc of diameter less than 2 yields a nonisotrivial miniversal Oka-map family, with pairwise nonbiholomorphic Oka fibres (Corollary 7.17). The full orbit space of bare biholomorphism classes is the punctured-disc curve $\mathcal{A}/2\mathbf{Z}$ (Corollary 7.15).

The even shifts admit a quadratic correction that makes their composition a strict $2\mathbf{Z}$-action. Together with vertical translations, they give a constructed analytic action groupoid $(\mathbf{C}^* \times 2\mathbf{Z}) \ltimes \mathcal{A}$ whose arrows exhaust the biholomorphisms between geometric fibres. Its quotient is $D^*_{M_0} \times B\mathbf{C}^*$ (Proposition C.4).

The paper is organized as follows. Section 2 constructs the period family and its fillings. Section 3 develops the cyclic bands and proves the relative Oka statement. The complete-lift and chart arguments for point and curve complements are in Section 4; Section 5 carries them through simultaneous blow-ups and records the smooth-curve conclusions. Section 6 computes deformations, while Section 7 recovers the intrinsic fibration, automorphisms and exact period. Appendix A records quotient tools, Appendix B verifies the toric and gluing steps, and Appendix C gives the centralizer calculation for exact period and the separate constructed-groupoid analysis. Appendix D identifies the ingredients of Alpöge's complex-six-sphere construction that enter this paper.

2. Period data and relative compactification

2.1. The period family and invariant direction. Let

$$B = \mathbf{P}^1, \qquad p_1 = 0, \quad p_2 = 1, \quad p_0 = \infty, \qquad B^\circ = B \setminus \{p_0, p_1, p_2\}.$$

Lemma 2.1. *There are a copy U of the upper half-plane, a properly discontinuous Fuchsian action of*

$$\Gamma = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1 \rangle,$$

and fixed points $z_1, z_2 \in U$ of g_1, g_2, respectively, such that $U/\Gamma \simeq B \setminus \{p_0\}$, with finite orbifold orders 3 and 4 at p_1, p_2. The element $g_0 = (g_1 g_2)^{-1}$ is a primitive parabolic generator of the cusp stabilizer. On

$$U^\circ = U \setminus (\Gamma z_1 \cup \Gamma z_2)$$

the action is free, and the orbit map

$$\varpi : U^\circ \longrightarrow B^\circ \tag{2.1}$$

is a holomorphic covering with deck group Γ.

Proof. Alpöge [2, Proposition 2.11(i)–(iii)] constructs this oriented $(3, 4, \infty)$ uniformization, including the primitive cusp generator in the stated convention. The finite stabilizers occur precisely on the two indicated orbits. Removing them gives the connected free locus, and the quotient map on that locus is the covering (2.1). □

For geometric orientation, Figure 1 shows a schematic patch of the corresponding triangle-group tessellation of U.

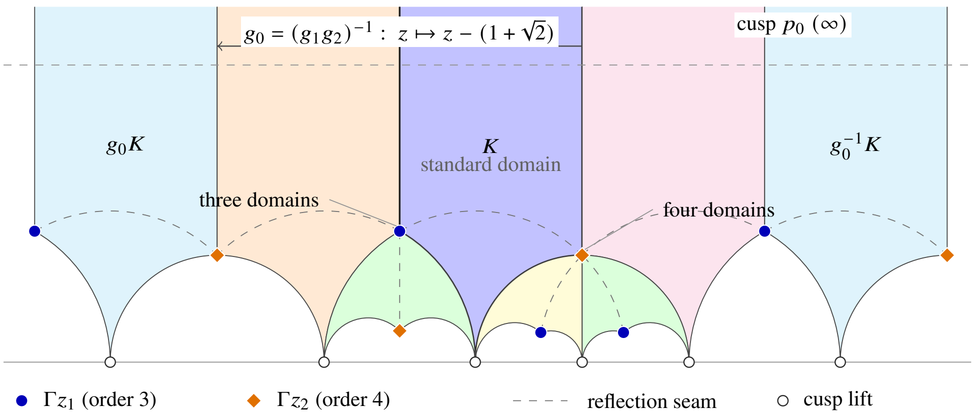

Figure 1. A patch of the $(3, 4, \infty)$ tessellation of U, after an upper-half-plane normalization. Each union of two adjacent $(\pi/3, \pi/4, 0)$ triangles is a fundamental domain for Γ. Three domains meet at every point of Γz_1, four at every point of Γz_2, and the ideal vertices lie over p_0. Only K and its translates $g_0K, g_0^{-1}K$ are labelled; the other domains are unmarked. With $L = 1 + \sqrt{2}$ and $s = z/L$, the indicated leftward translation satisfies $s(g_0 z) = s(z) - 1$, in agreement with the cusp coordinate $q = e^{2\pi i s}$ used below. Passing to U° removes the two elliptic orbits.

Let

$$\Lambda = \mathbf{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta} \rangle.$$

Write $\Lambda_{\mathbf{R}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$. The homological monodromy matrices in this ordered basis are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}. \tag{2.2}$$

Direct multiplication gives $A_1^3 = A_2^4 = I$. The displayed presentation of Γ therefore defines a representation $A : \Gamma \to \mathrm{GL}(\Lambda)$ by $A(g_j) = A_j$; its representation law is $A(gh) = A(g)A(h)$. The first row of each matrix in (2.2) is $(1, 0, 0, 0)$. Write $\gamma : \Lambda \to \mathbf{Z}$ for this invariant first-coordinate functional.

Proposition 2.2. *There are holomorphic functions*

$$\tau, \mu, \beta_0 : U \longrightarrow \mathbf{C}, \qquad \Im\tau > 0,$$

and holomorphic matrices $R_g : U \to \mathrm{GL}(2, \mathbf{C})$ satisfying

$$R_{gh}(z) = R_g(hz)R_h(z), \tag{2.3}$$

$$\Omega_a(gz) = R_g(z)\Omega_a(z)A(g)^{-1} \tag{2.4}$$

for every $g, h \in \Gamma$ and every $a \in \mathbf{C}$, where

$$\Omega_a(z) = \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta_0(z) + a & \mu(z) & 0 & 1 \end{pmatrix}. \tag{2.5}$$

The equations for all a follow from those for $a = 0$: the right block defining R_g is unchanged when a constant is added to β_0, and $\gamma A(g)^{-1} = \gamma$. Consequently the same equations hold for every $a \in \mathbf{C}$. The real-lattice condition is imposed separately in Proposition 2.3. Together with (2.3) and the representation law for A, this equivariance makes the orbifold descent an honest group action.

Set

$$\mathscr{D}_a(z) = \Im(\beta_0(z) + a) - \frac{6(\Im\mu(z))^2}{\Im\tau(z)}. \tag{2.6}$$

The function $\mathscr{D}_0$ is Γ-invariant, descends to $B \setminus \{p_0\}$, and tends to $-\infty$ as p_0 is approached. Its descended function is continuous at the two finite orbifold points; together with the cusp limit, this makes $\mathscr{D}_0$ bounded above.

Proof. In the ordered basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$ of (2.2), Alpöge [2, Theorem 3.4(i)–(v)] constructs τ, μ and a solution β_0, unique up to an additive constant, with the asserted cusp behaviour and the equivariance for $a = 0$. In the present generator convention their transformation laws are

$$\tau(g_1 z) = \frac{\tau(z) - 1}{\tau(z)}, \qquad \tau(g_2 z) = -\frac{1}{\tau(z)}, \qquad \tau(g_0 z) = \tau(z) - 1, \tag{2.7}$$

$$\mu(g_1 z) = \frac{1 - \mu(z)}{\tau(z)}, \qquad \mu(g_2 z) = 1 + \frac{\mu(z)}{\tau(z)}. \tag{2.8}$$

With $\varphi_1 = 2 - 6(1-\mu)^2/\tau$ and $\varphi_2 = -3 - 6\mu^2/\tau$, the remaining laws are

$$\beta_0(g_j z) = \beta_0(z) + \varphi_j(z), \qquad j = 1, 2. \tag{2.9}$$

The same theorem gives $R_{g_0} = I_2$ and, on the two generators,

$$R_{g_1} = \begin{pmatrix} -1/\tau & 0 \\ (1-\mu)/\tau & 1 \end{pmatrix}, \qquad R_{g_2} = \begin{pmatrix} 1/\tau & 0 \\ -\mu/\tau & 1 \end{pmatrix}.$$

In particular $R_g e_2 = e_2$ for all $g \in \Gamma$. Since $\Omega_a = \Omega_0 + a e_2 \gamma$ and $\gamma A(g)^{-1} = \gamma$, adding $a e_2 \gamma$ to the $a = 0$ equivariance gives (2.4) for every $a \in \mathbf{C}$; the same R_g still satisfies (2.3). Finally, Theorem 3.4(iv) gives the invariance, continuity at the finite orbifold points, and cusp limit of $\mathscr{D}_0$. These properties yield its upper bound on $B \setminus \{p_0\}$. □

Proposition 2.3. *Let*

$$M_0 = \sup_{z \in U} \mathscr{D}_0(z) < \infty.$$

Then

$$\mathscr{A} := \{a \in \mathbf{C} : \Im a < -M_0\} \tag{2.10}$$

is a nonempty open half-plane, and for every $a \in \mathscr{A}$ the four columns of $\Omega_a(z)$ are linearly independent over $\mathbf{R}$ for all $z \in U$.

Proof. Equation (2.6) gives

$$\mathscr{D}_a(z) = \mathscr{D}_0(z) + \Im a.$$

If $\Im a < -M_0$, then $\mathscr{D}_a(z) < 0$ for every z.

We now compute the real determinant. Write

$$Z_a = \begin{pmatrix} 6\mu & \tau \\ \beta_0 + a & \mu \end{pmatrix}, \qquad \Omega_a = [Z_a \mid I_2].$$

Using target real coordinates $(\Re z_1, \Im z_1, \Re z_2, \Im z_2)$, the real matrix of Ω_a is

$$\begin{pmatrix} \Re(6\mu) & \Re\tau & 1 & 0 \\ \Im(6\mu) & \Im\tau & 0 & 0 \\ \Re(\beta_0 + a) & \Re\mu & 0 & 1 \\ \Im(\beta_0 + a) & \Im\mu & 0 & 0 \end{pmatrix}.$$

Move the second row below the third row. This is one row transposition. The resulting matrix has block form

$$\begin{pmatrix} \Re Z_a & I_2 \\ \Im Z_a & 0 \end{pmatrix}.$$

Exchanging the two blocks of two columns has sign $(-1)^{2\cdot 2} = 1$ and gives a block upper-triangular matrix with diagonal blocks I_2 and $\Im Z_a$. Undoing the one row transposition therefore yields

$$\det_{\mathbf{R}} \Omega_a = -\det(\Im Z_a).$$

Since

$$\det(\Im Z_a) = 6(\Im\mu)^2 - \Im\tau\,\Im(\beta_0 + a),$$

we obtain

$$\det_{\mathbf{R}} \Omega_a = \Im\tau\,\mathscr{D}_a. \tag{2.11}$$

Here $\Im\tau > 0$ and $\mathscr{D}_a < 0$, so the determinant is nonzero. □

We call a parameter $a \in \mathscr{A}$ *admissible*.

The period variation has one invariant lattice direction. With $e_2 = (0, 1)^t$, the invariant functional γ has kernel

$$\Lambda_0 = \ker\gamma = \mathbf{Z}\langle \widehat{u}, \widehat{w}, \widehat{\delta}\rangle.$$

In particular, $\gamma A(g) = \gamma$ for every $g \in \Gamma$. For any two admissible parameters,

$$\Omega_a - \Omega_{a_*} = (a - a_*)e_2\gamma, \qquad \Omega_a\widehat{\delta} = e_2, \qquad R_g e_2 = e_2. \tag{2.12}$$

The last identity follows by differentiating period equivariance with respect to a and using $\gamma A(g)^{-1} = \gamma$. Translation by ue_2 therefore induces a compatible action

$$\Theta_{\exp(2\pi i u)}[z, \zeta_1, \zeta_2] = [z, \zeta_1, \zeta_2 + u]$$

on the quotients by Λ_0. Its extension across the finite and cusp fillings will be checked from their formulas. This direction will also represent the Kodaira–Spencer class in Subsection 6.4.

The equivariant period family now determines the smooth core over the punctured base. Its three missing fibres require separate local fillings.

2.2. **The smooth core of the relative family.**

Fix a disc $\Delta \Subset \mathscr{A}$. The parameter in this section is denoted $a \in \Delta$.

Here U is the orbifold universal cover, whereas $U^\circ \to B^\circ$ is the ordinary covering (2.1), with deck group Γ. The space U° is not simply connected: a small loop around a deleted orbit point has nonzero winding number about that point. The lattice quotient below uses this marked covering and does not require its total space to be simply connected.

Let Λ act on $U^\circ \times \mathbf{C}^2 \times \Delta$ by

$$\lambda \cdot (z, \zeta, a) = (z, \zeta + \Omega_a(z)\lambda, a). \tag{2.13}$$

Lemma 2.4. *The action (2.13) is free and properly discontinuous. Its quotient*

$$\widetilde{\mathcal{J}}_\Delta = (U^\circ \times \mathbf{C}^2 \times \Delta)/\Lambda$$

is a complex manifold, and the projection

$$\widetilde{p} : \widetilde{\mathcal{J}}_\Delta \longrightarrow U^\circ \times \Delta$$

is a proper holomorphic submersion with fibres complex two-tori.

Proof. Freeness follows from the injectivity of the real-linear map $\Omega_a(z) : \Lambda_{\mathbf{R}} \to \mathbf{C}^2$, proved in Proposition 2.3.

Let K_1, K_2 be compact subsets of $U^\circ \times \mathbf{C}^2 \times \Delta$. Their projections to $U^\circ \times \Delta$ lie in a compact set K. The smallest singular value of the real-linear map $\Omega_a(z)$ is a positive continuous function on K, hence has a positive lower bound c_K. If $\lambda K_1 \cap K_2 \neq \emptyset$, then for suitable points with the same (z, a) the difference of their ζ-coordinates is $\Omega_a(z)\lambda$. The compactness of K_1, K_2 bounds the norm of that difference by a constant C_K. Hence

$$c_K \|\lambda\| \leq \|\Omega_a(z)\lambda\| \leq C_K.$$

Only finitely many lattice vectors satisfy this inequality. Thus the action is properly discontinuous. The quotient is a complex manifold by Proposition A.5.

For properness, set

$$\Phi : (U^\circ \times \Delta) \times (\Lambda_{\mathbf{R}}/\Lambda) \longrightarrow \widetilde{\mathcal{J}}_\Delta, \qquad ((z, a), [x]) \longmapsto [z, \Omega_a(z)x, a].$$

The map is well defined and bijective. Its inverse is continuous because the inverse real-linear matrices $\Omega_a(z)^{-1}$ depend continuously on (z, a). Thus Φ is a homeomorphism over $U^\circ \times \Delta$. Since $\Lambda_{\mathbf{R}}/\Lambda$ is compact, the projection is proper. It is a holomorphic submersion because locally it is the quotient of the coordinate projection $U^\circ \times \mathbf{C}^2 \times \Delta \to U^\circ \times \Delta$ by translations. □

We next descend the lattice quotient by the deck group.

Equation (2.4) permits the action

$$g \cdot [z, \zeta, a] = [gz, R_g(z)\zeta, a] \tag{2.14}$$

on $\widetilde{\mathcal{J}}_\Delta$. It is well defined because

$$R_g(z)\Omega_a(z)\lambda = \Omega_a(gz)A(g)\lambda.$$

The group law follows from (2.3).

Proposition 2.5. *The action (2.14) is free and properly discontinuous, and the quotient*

$$\mathcal{J}_\Delta = \widetilde{\mathcal{J}}_\Delta/\Gamma$$

is a connected complex manifold. For every $a \in \Delta$, the parameter slice $(\mathrm{pr}_\Delta \circ p^\circ)^{-1}(a)$ is connected. The induced map

$$p^\circ : \mathcal{J}_\Delta \longrightarrow B^\circ \times \Delta$$

is a proper holomorphic submersion, and each fibre over a point of $B^\circ \times \{a\}$ is a complex two-torus with local period matrix Ω_a.

Proof. The action of Γ on U° is free and properly discontinuous. If an element fixes a point of $\widetilde{\mathcal{J}}_\Delta$, it fixes its base point in U°, hence is the identity. For compact subsets of $\widetilde{\mathcal{J}}_\Delta$, their projections to U° are compact; only finitely many group elements can make the projected compact sets meet. Thus the action is properly discontinuous. The quotient is a complex manifold by Proposition A.5. The homeomorphism in Lemma 2.4 identifies $\widetilde{\mathcal{J}}_\Delta$ topologically with $(U^\circ \times \Delta) \times (\Lambda_{\mathbf{R}}/\Lambda)$; all three factors are connected. Hence $\widetilde{\mathcal{J}}_\Delta$ is connected, and its continuous surjective quotient by Γ is connected. If $a \in \Delta$ is fixed, the same homeomorphism identifies the parameter slice upstairs with $U^\circ \times (\Lambda_{\mathbf{R}}/\Lambda)$, which is connected; its quotient by Γ is exactly $(\mathrm{pr}_\Delta \circ p^\circ)^{-1}(a)$ and is therefore connected.

Let $V \subset B^\circ$ be a simply connected relatively compact coordinate disc. Choose one connected lift $\widetilde{V} \subset U^\circ$. Every Γ-orbit over V meets $\widetilde{V}$ in exactly one point, because V is simply connected and $U^\circ \to B^\circ$ is a covering. Hence the restriction over $V \times \Delta$ is isomorphic to the lattice family over $\widetilde{V} \times \Delta$, so the restriction of p° is proper by Lemma 2.4. Properness is local on the target; a proof of the precise local-to-global statement used here is given in Lemma A.13. Submersivity follows in the same local chart. □

2.3. **Finite-point data and the order-3 and 4 fillings.** The finite local data are recorded here because every item below is used immediately in the affine cyclic fillings and their punctured identifications.

For $j = 1, 2$, set $m_1 = 3$ and $m_2 = 4$. The marked base has the following local uniformization data at p_j:

(F1)

$$\mathrm{Stab}_\Gamma(z_j) = \langle g_j \rangle, \qquad |\langle g_j \rangle| = m_j. \tag{2.15}$$

(F2) There is a g_j-invariant disc $\widetilde{D}_j \subset U$ centred at z_j such that

$$g\widetilde{D}_j \cap \widetilde{D}_j = \varnothing \qquad \text{for every } g \in \Gamma \setminus \langle g_j \rangle. \tag{2.16}$$

The two discs are chosen so that their images in B have disjoint closures.

(F3) There is a holomorphic coordinate s_j on $\widetilde{D}_j$ satisfying

$$s_j(g_j z) = \mathrm{e}^{-2\pi \mathrm{i}/m_j} s_j(z). \tag{2.17}$$

Verification of the finite local data. The stabilizer assertion follows from the triangle tessellation in Lemma 2.1. Proper discontinuity gives a small $\langle g_j \rangle$-invariant disc about z_j whose translates by other elements are disjoint; shrink the two discs so their coarse images have disjoint closures. The Cayley coordinate $s_j(z) = (z - z_j)/(z - \overline{z}_j)$ identifies U with the unit disc. Conjugating g_j by this coordinate gives a disc automorphism fixing the origin, hence a rotation. Our choice of inverse vertex rotations in Lemma 2.1 makes its multiplier $e^{-2\pi i/m_j}$, proving (2.17). □

Let

$$\widetilde{D}_j^* = \widetilde{D}_j \setminus \{z_j\}.$$

Because of (2.16), the restriction of the orbit map induces, by Proposition A.6, a biholomorphism

$$\widetilde{D}_j^*/\langle g_j \rangle \xrightarrow{\cong} D_j^* := \varpi(\widetilde{D}_j^*) \subset B^\circ. \tag{2.18}$$

Let D_j be the image of $\widetilde{D}_j$ in the coarse quotient $U/\Gamma \cong B \setminus \{p_0\}$; then $D_j = D_j^* \cup \{p_j\}$. The natural map

$$\widetilde{D}_j/\langle g_j \rangle \longrightarrow D_j$$

is bijective. Indeed, if $z, z' \in \widetilde{D}_j$ have the same image in U/Γ, then $z' = gz$ for some $g \in \Gamma$; the intersection $g\widetilde{D}_j \cap \widetilde{D}_j$ is nonempty, so (2.16) gives $g \in \langle g_j \rangle$. The map is open because the Γ-saturation of every $\langle g_j \rangle$-invariant open subset of $\widetilde{D}_j$ is open in U. It is therefore a homeomorphism onto D_j.

The invariant function

$$t_j = s_j^{m_j}$$

separates the $\langle g_j \rangle$-orbits: if $s_j(z)^{m_j} = s_j(z')^{m_j}$, then $s_j(z') = \mathrm{e}^{-2\pi \mathrm{i}k/m_j} s_j(z)$ for some integer k, so $z' = g_j^k z$. It follows that t_j is a holomorphic coordinate on the quotient disc, including at the image of z_j. Hence the preceding homeomorphism is biholomorphic.

The twist vectors are

$$v_1 = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}, \qquad v_2 = -\widehat{\gamma} - 3\widehat{u} + 3\widehat{w}. \tag{2.19}$$

Direct multiplication using (2.2) gives

$$A_1 v_1 = v_1, \qquad A_2 v_2 = v_2. \tag{2.20}$$

The invariant first-coordinate functional introduced above satisfies

$$\gamma(\widehat{\gamma}) = 1, \qquad \gamma(\widehat{u}) = \gamma(\widehat{w}) = \gamma(\widehat{\delta}) = 0.$$

Then

$$\gamma \circ A_j = \gamma, \qquad \gamma(v_1) = 1, \qquad \gamma(v_2) = -1. \tag{2.21}$$

Fix $j \in \{1, 2\}$. Let Λ act on $\widetilde{D}_j \times \mathbf{C}^2 \times \Delta$ by

$$\lambda \cdot (z, \zeta, a) = (z, \zeta + \Omega_a(z)\lambda, a),$$

and set

$$\mathcal{P}_{j,\Delta} = (\widetilde{D}_j \times \mathbf{C}^2 \times \Delta)/\Lambda.$$

The action is free because every real-linear map $\Omega_a(z)$ is injective. To prove proper discontinuity, let K_1, K_2 be compact subsets of the space before quotienting. On the compact projection of $K_1 \cup K_2$ to $\widetilde{D}_j \times \overline{\Delta}$, the smallest singular value of $\Omega_a(z)$ has a positive lower bound. If $\lambda K_1 \cap K_2 \neq \varnothing$, the difference of the two fibre coordinates is $\Omega_a(z)\lambda$; compactness bounds its norm, and the singular-value lower bound therefore bounds $\|\lambda\|$. Only finitely many lattice vectors occur. The quotient is a complex manifold by Proposition A.5.

The map

$$(\widetilde{D}_j \times \Delta) \times (\Lambda_{\mathbf{R}}/\Lambda) \longrightarrow \mathcal{P}_{j,\Delta}, \qquad ((z,a),[x]) \longmapsto [z, \Omega_a(z)x, a],$$

is a homeomorphism over $\widetilde{D}_j \times \Delta$: its inverse is induced by the continuously varying real-linear inverses $\Omega_a(z)^{-1}$. Since $\Lambda_{\mathbf{R}}/\Lambda$ is compact, the projection

$$\mathcal{P}_{j,\Delta} \longrightarrow \widetilde{D}_j \times \Delta$$

is proper. In a local quotient chart it is induced by the coordinate projection from $\widetilde{D}_j \times \mathbf{C}^2 \times \Delta$, so its differential is surjective and it is a holomorphic submersion.

The finite local data induce the following affine action.

Set

$$\widetilde{g}_j[z,\zeta,a] = \left[g_j z, R_{g_j}(z)\zeta + \Omega_a(g_j z)\frac{v_j}{m_j}, a\right]. \tag{2.22}$$

This formula is jointly holomorphic in (z,ζ,a). It is well defined on the lattice quotient: replacing ζ by $\zeta + \Omega_a(z)\lambda$ changes the target by

$$R_{g_j}(z)\Omega_a(z)\lambda = \Omega_a(g_j z)A_j\lambda,$$

which is an integral target period. The formula has a holomorphic inverse because its m_j-th power is the identity, as proved next.

Lemma 2.6. *The biholomorphism $\widetilde{g}_j$ has order exactly m_j, and the action of $\langle \widetilde{g}_j \rangle$ on $\mathcal{P}_{j,\Delta}$ is free.*

Proof. Use the real flat coordinate $x \in \Lambda_{\mathbf{R}}/\Lambda$ determined by $\zeta = \Omega_a(z)x$. Equation (2.4) transforms (2.22) into

$$(z,x,a) \longmapsto \left(g_j z, A_j x + \frac{v_j}{m_j}, a\right). \tag{2.23}$$

By (2.20), $A_j^r v_j = v_j$ for every r. Therefore the k-th power has fibre translation

$$\frac{1}{m_j}\sum_{r=0}^{k-1} A_j^r v_j = \frac{k v_j}{m_j}.$$

For $k = m_j$, the base rotation is the identity, $A_j^{m_j} = I$, and translation by $v_j \in \Lambda$ is the identity on the torus. Thus $\widetilde{g}_j^{m_j} = 1$. For $1 \le k < m_j$, the base rotation is nontrivial, so the k-th power cannot be the identity; the order is exactly m_j.

A fixed point of a nonidentity power must lie over z_j, because the base rotation has no other fixed point in $\widetilde{D}_j$. At z_j, the fixed point equation in the real torus is

$$(I - A_j^k)x \equiv \frac{k v_j}{m_j} \pmod{\Lambda}. \tag{2.24}$$

Apply the integral functional γ. Since $\gamma \circ A_j = \gamma$, the left side has γ-value zero modulo $\mathbf{Z}$. The right side has value

$$\frac{k}{3} \quad (j=1), \qquad -\frac{k}{4} \quad (j=2),$$

by (2.21). These numbers are not integers for $1 \le k < m_j$. Hence (2.24) has no solution. □

Proposition 2.7. *The quotient*

$$\mathcal{N}_{j,\Delta} = \mathcal{P}_{j,\Delta}/\langle \widetilde{g}_j \rangle$$

is a connected complex manifold, and every parameter slice of $\mathcal{N}_{j,\Delta} \to \Delta$ is connected. It has a proper holomorphic map

$$F_j : \mathcal{N}_{j,\Delta} \longrightarrow D_j \times \Delta,$$

and its projection to Δ is a holomorphic submersion. The fibre over $p_j \times \{a\}$ has multiplicity m_j, and its reduction is a smooth surface.

Proof. The group is finite and acts freely, so Proposition A.5 applies. The flat-coordinate homeomorphism above identifies $\mathcal{P}_{j,\Delta}$ with the product of the connected space $\widetilde{D}_j \times \Delta$ and the connected real torus $\Lambda_{\mathbf{R}}/\Lambda$. Thus $\mathcal{P}_{j,\Delta}$ is connected, and its continuous surjective finite quotient $\mathcal{N}_{j,\Delta}$ is connected. After fixing a, the same product description gives the connected space $\widetilde{D}_j \times (\Lambda_{\mathbf{R}}/\Lambda)$ upstairs; its finite quotient is the parameter slice of $\mathcal{N}_{j,\Delta}$ and is connected. Let

$$\rho_j : \widetilde{D}_j \times \Delta \longrightarrow D_j \times \Delta, \qquad (z,a) \longmapsto (s_j(z)^{m_j}, a).$$

The composite

$$\mathcal{P}_{j,\Delta} \longrightarrow \widetilde{D}_j \times \Delta \xrightarrow{\rho_j} D_j \times \Delta$$

is invariant under $\widetilde{g}_j$, because $s_j(g_j z)^{m_j} = s_j(z)^{m_j}$. It therefore descends to the holomorphic map F_j in the statement.

The map ρ_j is proper. Indeed, if $K \subset D_j \times \Delta$ is compact, then its projections are contained in a compact subdisc of D_j and a compact subset of Δ. Every solution of $s_j^{m_j} = t_j$ over that compact subdisc lies in a compact subdisc of $\widetilde{D}_j$, and $\rho_j^{-1}(K)$ is closed there. Hence $\rho_j^{-1}(K)$ is compact. Its fibres contain at most m_j points, so ρ_j is finite in the sense of appendix A. The inverse image of $\rho_j^{-1}(K)$ in $\mathcal{P}_{j,\Delta}$ is compact because the torus family is proper. Its image in the finite quotient is exactly $F_j^{-1}(K)$, hence is compact. Thus F_j is proper.

The parameter projection upstairs is a submersion and is invariant under the finite group. Since the quotient map is locally biholomorphic, the descended parameter projection is a submersion by Proposition A.5.

Finally, the coarse base coordinate is $t_j = s_j^{m_j}$. Its pullback upstairs vanishes to order m_j along the central torus. The quotient map of the total space is locally biholomorphic because the action is free. Upstairs, the reduced central divisor is the smooth torus family $s_j = 0$; its quotient by the free finite action is therefore smooth and is the reduction of $F_j^{-1}(p_j \times \Delta)$. Consequently this Cartier divisor has multiplicity m_j along its smooth reduction. □

On a simply connected sector of $\widetilde{D}_j^*$, choose a branch of $\log s_j$ and set

$$\sigma_{j,a}(z) = \frac{\log s_j(z)}{2\pi \mathrm{i}}\, \Omega_a(z) v_j. \tag{2.25}$$

The fibre translation is

$$h_j[z, \zeta, a] = [z, \zeta + \sigma_{j,a}(z), a].$$

If the branch of the logarithm changes by $2\pi \mathrm{i} n$, then $\sigma_{j,a}$ changes by $n\Omega_a(z)v_j$, an integral period. Hence the translation on the torus quotient is independent of the branch.

Lemma 2.8. *On $\mathcal{P}_{j,\Delta}|_{\widetilde{D}_j^* \times \Delta}$ one has*

$$h_j \widetilde{g}_j h_j^{-1} = \widetilde{g}_j^0, \qquad \widetilde{g}_j^0[z, \zeta, a] = [g_j z, R_{g_j}(z)\zeta, a].$$

Consequently there is a jointly holomorphic biholomorphism

$$G_j : \mathcal{N}_{j,\Delta}|_{D_j^* \times \Delta} \xrightarrow{\cong} \mathcal{J}_\Delta|_{D_j^* \times \Delta}. \tag{2.26}$$

Proof. From (2.17), on a chosen sector one has

$$\log s_j(g_j z) = \log s_j(z) - \frac{2\pi \mathrm{i}}{m_j}.$$

Using (2.4) and $A_j v_j = v_j$, we get

$$R_{g_j}(z)\Omega_a(z)v_j = \Omega_a(g_j z) A_j v_j = \Omega_a(g_j z) v_j.$$

Therefore

$$\begin{aligned} \sigma_{j,a}(g_j z) &= \frac{\log s_j(g_j z)}{2\pi \mathrm{i}}\, \Omega_a(g_j z) v_j \\ &= R_{g_j}(z)\sigma_{j,a}(z) - \Omega_a(g_j z)\frac{v_j}{m_j}. \end{aligned}$$

Substituting this identity into the definitions of h_j and $\widetilde{g}_j$ gives the displayed conjugation formula term by term.

We now identify the untwisted quotient with the actual restriction of the global core. Let

$$\widetilde{\mathcal{J}}^*_{j,\Delta} = \widetilde{\mathcal{J}}_\Delta|_{\widetilde{D}^*_j\times\Delta}.$$

The subgroup $\langle g_j\rangle$ acts on this space by the untwisted maps $\widetilde{g}^0_j$. If $g\widetilde{\mathcal{J}}^*_{j,\Delta}\cap\widetilde{\mathcal{J}}^*_{j,\Delta}\neq\varnothing$, then the projections to U° give $g\widetilde{D}^*_j\cap\widetilde{D}^*_j\neq\varnothing$. Equation (2.16) therefore implies $g\in\langle g_j\rangle$. Applying Proposition A.6 to the Γ-action on $\widetilde{\mathcal{J}}_\Delta$ gives a biholomorphism

$$\widetilde{\mathcal{J}}^*_{j,\Delta}/\langle g_j\rangle \xrightarrow{\cong} \mathcal{J}_\Delta|_{D^*_j\times\Delta}.$$

The affine quotient of the same pulled-back family is $\mathcal{N}_{j,\Delta}|_{D^*_j\times\Delta}$. Hence the conjugation descends to (2.26). Every formula is holomorphic in (z,a), and branch changes alter the translation by an integral period, so the descended map is jointly holomorphic and single-valued. □

2.4. **Cusp data, toric model, and relative filling.** The cusp quotient requires a Hausdorff countable-fan model, a proper lattice action that remains controlled as $q\to 0$, and a punctured identification with the smooth core. The statements and estimates are proved below; full chartwise verifications are collected in appendix B.

We choose a simply connected cusp neighbourhood $\mathcal{H}_0\subset U^\circ$ and a biholomorphic coordinate

$$s:\mathcal{H}_0\xrightarrow{\cong}\{w\in\mathbf{C}:\Im w>R_0\}$$

for some real number R_0, satisfying

$$s(g_0z)=s(z)-1.$$

The neighbourhood and coordinate can be chosen to obey

$$\operatorname{Stab}_\Gamma(\mathcal{H}_0)=\langle g_0\rangle, \tag{2.27}$$

$$g\mathcal{H}_0\cap\mathcal{H}_0=\varnothing\qquad (g\in\Gamma\setminus\langle g_0\rangle). \tag{2.28}$$

Indeed, a sufficiently small horodisc at the parabolic fixed point of g_0 is precisely invariant under $\langle g_0\rangle$ by the discrete triangle-group action in Lemma 2.1. The function $s=\tau-h(t_c)$ constructed in Proposition 2.2 satisfies $e^{2\pi is}=t_c$. On a smaller cusp disc, h is bounded, so a high enough half-plane in the s coordinate lies inside that precisely invariant horodisc. Its inverse image is $\mathcal{H}_0$, proving (2.27) and (2.28). Here the first equality means that the setwise stabilizer of the cusp neighbourhood is the infinite cyclic group generated by the primitive parabolic element g_0. After increasing R_0 and replacing $\mathcal{H}_0$ by the inverse image of the corresponding higher half-plane if necessary, set

$$q=\mathrm{e}^{2\pi\mathrm{i}s},\qquad \varepsilon_0=\mathrm{e}^{-2\pi R_0}.$$

The function q then identifies the quotient $\mathcal{H}_0/\langle g_0\rangle$ biholomorphically with the punctured disc

$$D^*_0=\{q\in\mathbf{C}:0<|q|<\varepsilon_0\}, \tag{2.29}$$

and the induced map $D^*_0\to B^\circ$ is the restriction of the global orbit map (2.1). Equivalently, this identification follows from (2.28) and Proposition A.6. Let

$$D_0=D^*_0\cup\{p_0\}\cong\{|q|<\varepsilon_0\}.$$

The three discs D_0,D_1,D_2 are chosen with pairwise disjoint closures.

In the splitting

$$\Lambda_{\mathrm{tor}}=\mathbf{Z}\langle\widehat{w},\widehat{\delta}\rangle,\qquad \overline{\Lambda}=\Lambda/\Lambda_{\mathrm{tor}}=\mathbf{Z}\langle\widehat{\gamma},\widehat{u}\rangle,$$

write $\Omega_a=[Z_a\mid I_2]$. The nonconstant 2×2 period block has the form

$$Z_a=sB_0+C_a(q),\qquad B_0=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\qquad C_a(q)=C_0(q)+aE_{21}, \tag{2.30}$$

where C_0 is holomorphic on a disc D_{ε_0} about $q=0$ and E_{21} has a single 1 in row 2, column 1. To verify the decomposition, let $b=\beta_0+\tau$. The proof of Proposition 2.2 shows that b and μ are g_0-invariant and holomorphic in $q=t_c$ at 0, and that $\tau=s+h(q)$. Hence the displayed period matrix gives

$$C_0(q)=\begin{pmatrix}6\mu(q)&h(q)\\b(q)-h(q)&\mu(q)\end{pmatrix},$$

which is holomorphic at $q = 0$. The same proposition computes $R_{g_0} = I_2$, and the additive parameter contributes aE_{21}. Since $q(g_0 z) = q(z)$ and $s(g_0 z) = s(z) - 1$, one has

$$Z_a(g_0 z) = Z_a(z) - B_0. \tag{2.31}$$

These identities will be used in the punctured identification.

We recall the toric notation used for the periodic fan.

Let N be a lattice, let $M = \mathrm{Hom}(N, \mathbf{Z})$, and write $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$. A *rational polyhedral cone* is a strongly convex cone generated by finitely many elements of N. A *rational fan* Σ is a collection of rational polyhedral cones such that every face of a cone in Σ belongs to Σ and the intersection of two cones in Σ is a face of each. No finiteness or ambient local-finiteness condition is included in this definition. For $\sigma \in \Sigma$, set

$$U_\sigma = \mathrm{Spec}\mathbf{C}[\sigma^\vee \cap M]$$

and use the same notation for its complex analytification.

For finite subfans we use the standard face-chart embeddings, separated gluing, orbit–cone correspondence, and star-fan description of toric divisors from Fulton [6, Sections 1.3–1.4, 2.1, 3.1 and 3.3]. For the specific periodic fan below, the countable gluing and its torus action are established by Alpöge [2, Lemmas 4.2–4.3]. We identify that fan with the present coordinates next.

Let $N = \mathbf{Z}^2$ with standard basis e_1, e_2, and triangulate $N_{\mathbf{R}} = \mathbf{R}^2$ by the triangles

$$\begin{aligned} P_v^+ &= \mathrm{conv}\{v, v + e_1, v + e_2\}, \\ P_v^- &= \mathrm{conv}\{v + e_1, v + e_2, v + e_1 + e_2\}, \qquad v \in \mathbf{Z}^2. \end{aligned}$$

Let $N' = N \oplus \mathbf{Z}$. For every vertex, edge, or triangle P of the triangulation, set

$$\sigma_P = \mathrm{cone}(P \times \{1\}) \subset N'_{\mathbf{R}},$$

and set

$$\Sigma = \{0\} \cup \{\sigma_P : P \text{ is a cell of the triangulation}\}.$$

Proposition 2.9. *The collection Σ is a countable rational fan. Every cone is generated by part of a basis of N'. For each vertex $v \in \mathbf{Z}^2$, the star fan of the ray $\mathbf{R}_{\geq 0}(v, 1)$ is the finite complete hexagonal fan. Consequently the countable toric space*

$$Y = Y_\Sigma$$

is a smooth Hausdorff second-countable complex threefold, and the toric divisor E_v corresponding to the ray through $(v, 1)$ is compact.

Proof. The two families of triangles above are exactly the periodic A_2-triangulation used by Alpöge [2, Lemma 4.2(i)–(iv)], under the identification of its height-one lattice with $N' = \mathbf{Z}^2 \oplus \mathbf{Z}$. In particular, that lemma proves the fan axioms, unimodularity, Hausdorff and second-countable toric gluing, and the complete hexagonal star fan of each ray through $(v, 1)$. The source's toric threefold and divisors are therefore precisely Y_Σ and E_v in the present notation. □

Let $h = (0, 0, 1) \in M' = \mathrm{Hom}(N', \mathbf{Z})$. Since h is nonnegative on every cone, the character χ^h is holomorphic on every affine toric chart and the chartwise functions glue to

$$q : Y \longrightarrow \mathbf{C}.$$

If a maximal cone is generated by the rays through $(v_0, 1), (v_1, 1), (v_2, 1)$ and m_0, m_1, m_2 is the dual basis, then $h = m_0 + m_1 + m_2$. In the affine coordinates $z_i = \chi^{m_i}$ one has

$$q = z_0 z_1 z_2. \tag{2.32}$$

The divisor $q^{-1}(0)$ is therefore reduced with simple normal crossings. The component E_v corresponding to the ray through $(v, 1)$ has the compact hexagonal star fan described in Proposition 2.9. We shall use repeatedly that E_0 is a compact smooth toric surface.

All matrix norms in the definition below and in the cusp analysis are the operator norms induced by the standard Euclidean norms on the relevant real vector spaces.

Definition 2.10. Let $\Delta \Subset \mathcal{A}$ be a relatively compact parameter disc. A number $\varepsilon > 0$ is called a *valid cusp cutoff for* Δ if there exists a number ε_1 such that

(V1)

$$0 < \varepsilon < \varepsilon_1 < \min\{1, \varepsilon_0\};$$

(V2) with

$$M_C(\Delta,\varepsilon_1) = \sup_{\substack{|q|\le\varepsilon_1\\ a\in\overline{\Delta}}} \left\|-2\pi\Im C_a(q)\right\|,$$

one has

$$\frac{\|B_0^{-1}\|M_C(\Delta,\varepsilon_1)}{|\log\varepsilon|} \le \frac{1}{2}; \tag{2.33}$$

(V3) the closed subdisc $\overline{D}_\varepsilon = \{|q| \le \varepsilon\}$, viewed inside the original cusp coordinate disc $D_{\varepsilon_0} \cup \{p_0\} \subset B$, is disjoint from $\overline{D}_1 \cup \overline{D}_2$.

A number ε_1 satisfying these conditions is called a *witness* for the validity of ε.

Lemma 2.11. *For every relatively compact parameter disc $\Delta \Subset \mathcal{A}$, valid cusp cutoffs exist. If ε is valid for Δ and $0 < \varepsilon' < \varepsilon$, then ε' is valid for Δ with the same witness.*

Proof. Choose any $0 < \varepsilon_1 < \min\{1, \varepsilon_0\}$. Since C_0 is holomorphic on D_{ε_0}, the function $(q, a) \mapsto C_a(q) = C_0(q) + aE_{21}$ is continuous on the compact set $\overline{D}_{\varepsilon_1} \times \overline{\Delta}$. Thus $M_C(\Delta, \varepsilon_1) < \infty$. As $\varepsilon \downarrow 0$, the denominator $|\log\varepsilon|$ tends to infinity, so condition (V2) holds for all sufficiently small positive $\varepsilon < \varepsilon_1$. The original cusp disc was chosen with closure disjoint from the two finite discs; hence the same is true for every sufficiently small closed subdisc. This proves existence.

If $0 < \varepsilon' < \varepsilon$, then $|\log\varepsilon'| > |\log\varepsilon|$, so the left side of (2.33) can only decrease. The smaller closed cusp disc is contained in the larger one, so condition (V3) is also preserved. The same ε_1 is therefore a witness for ε'. ◻

The cusp is the only place where an infinite group acts on a non-quasi-compact toric space. We give separate proofs of joint holomorphic extension, freeness, proper discontinuity, and properness of the quotient map.

Fix a valid cusp cutoff ε for Δ in the sense of Definition 2.10, and choose one witness ε_1. Set

$$M_C = M_C(\Delta,\varepsilon_1) = \sup_{\substack{|q|\le\varepsilon_1\\ a\in\overline{\Delta}}} \left\|-2\pi\Im C_a(q)\right\| < \infty. \tag{2.34}$$

By condition (V2) in Definition 2.10,

$$\frac{\|B_0^{-1}\|M_C}{|\log\varepsilon|} \le \frac{1}{2}. \tag{2.35}$$

We now restrict every cusp object to this radius and write

$$D_0 = D_\varepsilon = \{q \in \mathbf{C} : |q| < \varepsilon\}, \qquad D_0^* = D_\varepsilon \setminus \{0\}, \qquad Y_\varepsilon = q^{-1}(D_\varepsilon).$$

On the uniformizing side take the smaller horodisc

$$\mathcal{H}_{0,\varepsilon} = \{z \in \mathcal{H}_0 : 0 < |q(z)| < \varepsilon\}. \tag{2.36}$$

Since $|q(z)| = \exp(-2\pi\Im s(z))$, this is a connected horodisc. It is $\langle g_0\rangle$-invariant, and the quotient map q induces a biholomorphism

$$\mathcal{H}_{0,\varepsilon}/\langle g_0\rangle \xrightarrow{\cong} D_0^*. \tag{2.37}$$

Moreover, if $g\mathcal{H}_{0,\varepsilon} \cap \mathcal{H}_{0,\varepsilon} \ne \varnothing$, then $g\mathcal{H}_0 \cap \mathcal{H}_0 \ne \varnothing$; by (2.28), this forces $g \in \langle g_0\rangle$. Thus the isolated-subgroup property survives the shrinking. Condition (V3) in Definition 2.10 gives $\overline{D}_0 \cap (\overline{D}_1 \cup \overline{D}_2) = \varnothing$.

Identify $\overline{\Lambda}$ with $\mathbf{Z}^2$ in the basis $(\widehat{\gamma}, \widehat{u})$. For $\lambda \in \mathbf{Z}^2$, the lattice automorphism

$$\phi_\lambda : N' \longrightarrow N', \qquad (y, r) \longmapsto (y + rB_0\lambda, r) \tag{2.38}$$

preserves the fan Σ. Let Φ_λ be the induced toric automorphism of Y. On the dense torus $(\mathbf{C}^*)^2 \times \mathbf{C}^*$ it is

$$\Phi_\lambda(x, q) = (q^{B_0\lambda}x, q).$$

For $(q, a) \in D_\varepsilon \times \Delta$, set

$$c_{a,\lambda}(q) = \exp(2\pi\mathrm{i}C_a(q)\lambda) \in (\mathbf{C}^*)^2, \tag{2.39}$$

where exponentiation is componentwise.

Proposition 2.12. *For every $\lambda \in \mathbf{Z}^2$, the formula*

$$\Psi_\lambda(y, a) = \big(c_{a,\lambda}(q(y)) \cdot \Phi_\lambda(y), a\big) \tag{2.40}$$

defines a biholomorphism of $Y_\varepsilon \times \Delta$. The dependence on (y, a) is holomorphic, and

$$\Psi_\lambda \Psi_\mu = \Psi_{\lambda+\mu}.$$

On the dense torus,

$$\Psi_\lambda(x, q, a) = \big(c_{a,\lambda}(q) q^{B_0\lambda} x, q, a\big). \tag{2.41}$$

Proof. Alpöge [2, Lemma 4.2(i)] gives the holomorphic torus multiplication map $(\mathbf{C}^*)^2 \times Y \to Y$ for this fan. The map

$$(y, a) \longmapsto c_{a,\lambda}(q(y))$$

is holomorphic because $C_a(q) = C_0(q) + aE_{21}$ is jointly holomorphic in (q, a). Therefore torus translation by this point is jointly holomorphic on $Y \times \Delta$. Composing it with the toric biholomorphism Φ_λ gives (2.40).

Both the vertical torus action and Φ_λ preserve q. Hence Ψ_λ preserves $Y_\varepsilon \times \Delta$.

The shear (2.38) restricts to the identity on the vertical sublattice $N \oplus \{0\}$. Hence Φ_λ commutes with the vertical torus action. Moreover,

$$c_{a,\lambda}(q) c_{a,\mu}(q) = c_{a,\lambda+\mu}(q)$$

because $C_a(q)$ is linear in the lattice vector. The shears add: $\phi_\lambda \phi_\mu = \phi_{\lambda+\mu}$. These identities prove the group law. Taking $\mu = -\lambda$ shows that $\Psi_{-\lambda}$ is the inverse of Ψ_λ. Formula (2.41) is the corresponding expression on the dense torus. □

Thus the action extends jointly holomorphically across the toric boundary.

For $0 < |q| < \varepsilon$, set

$$R_a(q) = -2\pi \Im C_a(q), \qquad B_{q,a} = B_0 + \frac{R_a(q)}{\log|q|}. \tag{2.42}$$

Here $R_a(q)$ is a real 2×2 matrix.

Lemma 2.13. *For the radius ε chosen in (2.35), there is a constant $K > 0$ such that*

$$\|B_{q,a}\| + \|B_{q,a}^{-1}\| \le K$$

for every $0 < |q| < \varepsilon$ and every $a \in \overline{\Delta}$.

Proof. For $0 < |q| < \varepsilon$ and $a \in \overline{\Delta}$, (2.34) gives

$$\|R_a(q)\| \le M_C.$$

Because $0 < |q| < \varepsilon < 1$, one has $|\log|q|| \ge |\log \varepsilon|$. Therefore

$$\left\| B_0^{-1} \frac{R_a(q)}{\log|q|} \right\| \le \frac{\|B_0^{-1}\| M_C}{|\log \varepsilon|} \le \frac{1}{2}$$

by (2.35). Set

$$E_{q,a} = B_0^{-1} \frac{R_a(q)}{\log|q|}.$$

Then

$$B_{q,a} = B_0(I + E_{q,a}), \qquad \|E_{q,a}\| \le \frac{1}{2}.$$

The Neumann series converges absolutely and gives

$$(I + E_{q,a})^{-1} = \sum_{n=0}^{\infty} (-E_{q,a})^n, \qquad \|(I + E_{q,a})^{-1}\| \le \sum_{n=0}^{\infty} 2^{-n} = 2.$$

Consequently

$$\|B_{q,a}\| \le \|B_0\| + \frac{M_C}{|\log \varepsilon|}$$

and

$$\|B_{q,a}^{-1}\| = \|(I + E_{q,a})^{-1} B_0^{-1}\| \le 2\|B_0^{-1}\|.$$

Taking the sum of the two displayed upper bounds gives the required constant K. □

On the dense torus use the logarithmic modulus coordinate

$$y(x,q) = \frac{\log|x|}{\log|q|} \in \mathbf{R}^2. \tag{2.43}$$

Taking absolute values in (2.41) gives

$$y(\Psi_\lambda(x,q,a)) = y(x,q) + B_{q,a}\lambda. \tag{2.44}$$

The next lemma is the point that controls mixed sequences with $q \neq 0$ and $q \to 0$.

Lemma 2.14. *Let $K \subset Y_\varepsilon$ be compact. There exists $M_K > 0$ such that*

$$\|y(x,q)\| \leq M_K$$

for every $(x,q) \in K$ lying in the dense torus with $q \neq 0$.

Proof. For every point of K, choose a relatively compact open neighbourhood whose closure lies in an affine chart associated with a maximal triangle. A finite number of these smaller neighbourhoods covers K. Their intersections with K have compact closures inside the corresponding affine charts. It is therefore enough to prove the assertion on one such compact chart piece.

Let $P = \operatorname{conv}\{v_0, v_1, v_2\}$ be the corresponding maximal triangle. Its cone is generated by $(v_i, 1)$, which form a basis of N'. Let m_i be the dual basis and set $z_i = \chi^{m_i}$. The affine barycentric functions are

$$\ell_i(y) = \langle m_i, (y,1)\rangle.$$

They satisfy

$$\ell_i(v_j) = \delta_{ij}, \qquad \ell_0 + \ell_1 + \ell_2 = 1, \qquad y = \sum_{i=0}^{2} \ell_i(y) v_i.$$

For a dense-torus point, the character formula gives

$$\log|z_i| = \ell_i(y)\log|q|. \tag{2.45}$$

On the chosen compact chart piece, choose $R \geq 1$ such that $|z_i| \leq R$ for all i. Split into two ranges. Fix $0 < \eta < \varepsilon$. If $|q| \geq \eta$, then $q = z_0 z_1 z_2$ and $|z_i| \leq R$ imply

$$|z_i| \geq \frac{\eta}{R^2} \qquad (i = 0,1,2).$$

Thus every $\log|z_i|$ is bounded above and below. Moreover $\eta \leq |q| < \varepsilon < 1$, so $\log|q|$ is bounded away from zero. Equation (2.45) then bounds all $\ell_i(y)$ and hence bounds y. If $0 < |q| < \eta$, equation (2.45) and $|z_i| \leq R$ give

$$\ell_i(y) = \frac{\log|z_i|}{\log|q|} \geq -\frac{\log R}{|\log|q||}.$$

Choose η so small that the right side is at least -1. Since the three ℓ_i sum to 1, each is then at most 3. The vertices v_i are fixed, so

$$\|y\| = \left\| \sum_i \ell_i(y) v_i \right\|$$

is uniformly bounded. Taking the maximum over the finite chart cover proves the lemma. □

Proposition 2.15. *The $\mathbf{Z}^2$-action (2.40) on $Y_\varepsilon \times \Delta$ is free.*

Proof. Suppose first that $q \neq 0$ and $\Psi_\lambda(x,q,a) = (x,q,a)$. Equation (2.44) gives $B_{q,a}\lambda = 0$. By Lemma 2.13, $\lambda = 0$.

Now let $q = 0$. Torus orbits in the central fibre correspond to cells P of the height-one triangulation. The toric automorphism Φ_λ sends the orbit of P to the orbit of $P + B_0\lambda$. The additional torus translation $c_{a,\lambda}(0)$ preserves each torus orbit. Therefore a fixed point would require

$$P + B_0\lambda = P.$$

Every cell P is bounded, so it is invariant under a translation only when the translation vector is zero. Since B_0 is invertible, $\lambda = 0$. □

Proposition 2.16. *The $\mathbf{Z}^2$-action on $Y_\varepsilon \times \Delta$ is properly discontinuous.*

Proof. Let $K_1, K_2 \subset Y_\varepsilon \times \Delta$ be compact. We must show that only finitely many $\lambda \in \mathbf{Z}^2$ satisfy

$$\Psi_\lambda(K_1) \cap K_2 \neq \varnothing.$$

First consider intersections represented by points with $q \neq 0$. The projections of K_1, K_2 to Y_ε are compact, so Lemma 2.14 gives constants M_1, M_2 bounding the logarithmic coordinates of the source and target points. If $p \in K_1$ and $\Psi_\lambda(p) \in K_2$, then (2.44) gives

$$\lambda = B_{q,a}^{-1}(y(\Psi_\lambda p) - y(p)).$$

By Lemma 2.13,

$$\|\lambda\| \leq K(M_1 + M_2).$$

Only finitely many integer vectors satisfy this bound. Notice that this argument remains valid for sequences $q \neq 0$ tending to 0; that is exactly the content of Lemma 2.14.

It remains to consider intersections lying in $q = 0$. For $i = 1, 2$, let

$$K_i^0 = \mathrm{pr}_{Y_\varepsilon}\big(K_i \cap (q^{-1}(0) \times \Delta)\big).$$

Each K_i^0 is compact. A finite collection of affine toric charts covers each K_i^0, and every chosen chart contains only finitely many torus orbits, corresponding to the faces of its cone. Hence there are finite sets of cells $\mathcal{S}_1, \mathcal{S}_2$ such that K_i^0 meets only the orbits associated with cells in $\mathcal{S}_i$. If an orbit meeting K_1^0 is sent to one meeting K_2^0, then

$$P + B_0\lambda = P'$$

for some $P \in \mathcal{S}_1$ and $P' \in \mathcal{S}_2$. For each pair (P, P') there is at most one λ, because B_0 is invertible. Thus only finitely many λ occur on the central fibre as well. □

Lemma 2.17. *Let $K_y \subset \mathbf{R}^2$ be bounded and let $0 < r < \varepsilon$. The closure in Y of*

$$\mathcal{S}(K_y, r) = \{(x, q) \in (\mathbf{C}^*)^2 \times \mathbf{C}^* : |q| \leq r,\ y(x, q) \in K_y\}$$

is compact.

Proof. It suffices to replace K_y by $\overline{K_y}$, since $\mathcal{S}(K_y, r) \subset \mathcal{S}(\overline{K_y}, r)$. We may therefore assume that K_y is compact. It meets only finitely many triangles of the locally finite triangulation. For each $y \in K_y$, choose a triangle $P = \mathrm{conv}\{v_0, v_1, v_2\}$ containing y. In the corresponding affine chart, the barycentric functions satisfy $\ell_i(y) \geq 0$ and $\sum_i \ell_i(y) = 1$. Equation (2.45) gives

$$|z_i| = |q|^{\ell_i(y)} \leq 1.$$

Thus the part of $\mathcal{S}(K_y, r)$ associated with P lies in the closed unit polydisc of the chart $U_P \cong \mathbf{C}^3$. Only finitely many triangles are needed, so the closure is contained in a finite union of compact closed polydiscs. It is closed in that finite union and is therefore compact. □

Proposition 2.18. *Let $K \subset D_\varepsilon \times \Delta$ be compact. There exists a compact set*

$$C_K \subset (q, a)^{-1}(K) \subset Y_\varepsilon \times \Delta$$

meeting every $\mathbf{Z}^2$-orbit over K.

Proof. Choose $r < \varepsilon$ and a compact set $K_a \Subset \Delta$ such that $K \subset \overline{D}_r \times K_a$. For $0 < |q| \leq r$ and $a \in K_a$, set

$$\mathcal{P}_{q,a} = B_{q,a}([-1/2, 1/2]^2).$$

By Lemma 2.13, all these parallelograms lie in one bounded set $K_y \subset \mathbf{R}^2$.

Let (x, q, a) have $q \neq 0$. Choose $\lambda \in \mathbf{Z}^2$ such that

$$B_{q,a}^{-1} y(x, q) + \lambda \in [-1/2, 1/2]^2.$$

Then (2.44) gives

$$y(\Psi_\lambda(x, q, a)) = B_{q,a}\big(B_{q,a}^{-1} y(x, q) + \lambda\big) \in K_y.$$

Therefore the compact set

$$C^* = \overline{\mathcal{S}(K_y, r)} \times K_a$$

meets every orbit with $q \neq 0$ over the larger box $\overline{D}_r \times K_a$, by Lemma 2.17.

Now consider $q = 0$. Let a point lie in the orbit corresponding to a cell P. Choose a vertex v of P. Since $B_0 \in \mathrm{GL}(2, \mathbf{Z})$, there is a unique $\lambda \in \mathbf{Z}^2$ with $B_0\lambda = -v$. The translated cell $P + B_0\lambda$ contains 0. Hence the translated torus orbit is contained in the component E_0 of the central fibre. The torus factor $c_{a,\lambda}(0)$ preserves that orbit and therefore does not alter this conclusion. Thus the compact set $E_0 \times K_a$ meets every central-fibre orbit.

Set

$$C_{\text{box}} = C^* \cup (E_0 \times K_a).$$

It is compact and meets every orbit over $\overline{D}_r \times K_a$. Because the group action preserves (q, a), the set

$$C_K = C_{\text{box}} \cap (q, a)^{-1}(K)$$

is compact and still meets every orbit over K. □

Lemma 2.19. *Let*

$$D_Y = q^{-1}(0) \subset Y.$$

Then D_Y is reduced, and its normalization is the disjoint union

$$\nu_Y : \coprod_{v \in \mathbf{Z}^2} E_v \longrightarrow D_Y \tag{2.46}$$

of its smooth irreducible toric components.

In a maximal toric chart the central equation is $q = z_0 z_1 z_2$. Hence its branches are smooth coordinate hyperplanes, and their local normalizations glue precisely to the globally defined components E_v. The complete verification is given in Appendix B.

Proposition 2.20. *Fix $a \in \Delta$ and set*

$$\widetilde{D}_Y = \coprod_{v \in \mathbf{Z}^2} E_v.$$

The action of $\mathbf{Z}^2$ on D_Y lifts to a free properly discontinuous action by biholomorphisms on $\widetilde{D}_Y$. Its analytic orbit space is biholomorphic to E_0.

The lifted action sends E_v to $E_{v+B_0\lambda}$. Since $B_0 \in \mathrm{GL}(2, \mathbf{Z})$, it is free and transitive on the set of components, while compact subsets of the disjoint union meet only finitely many components. Thus the quotient has the single representative component E_0. The complete verification is given in Appendix B.

Theorem 2.21. *The quotient*

$$\mathcal{N}_{0,\Delta} = (Y_\varepsilon \times \Delta)/\mathbf{Z}^2$$

is a connected Hausdorff second-countable complex manifold, and every parameter slice of $\mathcal{N}_{0,\Delta} \to \Delta$ is connected. The induced map

$$F_0 : \mathcal{N}_{0,\Delta} \longrightarrow D_\varepsilon \times \Delta$$

is proper, and the projection to Δ is a holomorphic submersion. The inverse image

$$W_\Delta := F_0^{-1}(\{0\} \times \Delta)$$

is a reduced normal-crossing divisor in $\mathcal{N}_{0,\Delta}$. For every $a \in \Delta$, the fibre

$$W_a := F_0^{-1}(0, a)$$

is a compact, reduced, irreducible, nonnormal surface with normal-crossing singularities, and its normalization is isomorphic to the compact toric surface E_0.

Proof. Freeness and proper discontinuity are Propositions 2.15 and 2.16; hence the quotient is a complex manifold by Proposition A.5. The dense torus of Y meets Y_ε in

$$(\mathbf{C}^*)^2 \times D_\varepsilon^*,$$

which is connected. This subset is dense in Y_ε, so Y_ε is connected as the closure of a connected subset. The product $Y_\varepsilon \times \Delta$ is therefore connected, and its continuous surjective quotient $\mathcal{N}_{0,\Delta}$ is connected. For fixed $a \in \Delta$, the slice upstairs is the connected space Y_ε, and its quotient by the restricted $\mathbf{Z}^2$-action is the corresponding parameter slice of $\mathcal{N}_{0,\Delta}$; hence that slice is connected as well. Properness follows from Proposition 2.18 and Lemma A.7. The parameter projection upstairs is the ordinary projection from $Y_\varepsilon \times \Delta$, hence is a submersion; the descended map is a submersion by Proposition A.5.

In every maximal toric chart, the equation $q = 0$ is $z_0 z_1 z_2 = 0$ by (2.32). The quotient map is locally biholomorphic, so the same equation describes

$$W_\Delta = F_0^{-1}(\{0\} \times \Delta).$$

Hence W_Δ is a reduced normal-crossing divisor in the four-dimensional manifold $\mathcal{N}_{0,\Delta}$. Fix $a \in \Delta$. Intersecting these local charts with the parameter slice a = constant shows that $W_a = F_0^{-1}(0, a)$ is a reduced normal-crossing complex surface. It is compact because F_0 is proper. Set $D_Y = q^{-1}(0)$.

We now identify the normalization of W_a without appealing only to the component count. By Lemma 2.19, the normalization of D_Y is

$$\widetilde{D}_Y = \coprod_{\nu\in\mathbf{Z}^2} E_\nu.$$

By Proposition 2.20, the lifted action on $\widetilde{D}_Y$ is free and properly discontinuous, its analytic orbit space is defined by the quotient theorem, and there is a biholomorphism

$$\widetilde{D}_Y/\mathbf{Z}^2 \xrightarrow{\cong} E_0. \tag{2.47}$$

Let $p : D_Y \to W_a$ be the restriction of the quotient map and let $\nu_Y : \widetilde{D}_Y \to D_Y$ be (2.46). The composite $p \circ \nu_Y$ is $\mathbf{Z}^2$-invariant and therefore factors through a holomorphic map

$$\nu_a : E_0 \longrightarrow W_a. \tag{2.48}$$

This map is surjective because every point of D_Y lies on at least one component E_ν, and that component has a translate equal to E_0.

The map ν_a is proper because E_0 is compact and W_a is Hausdorff. Its fibres are finite. To see this, fix $w \in W_a$ and choose $x \in p^{-1}(w)$. The normalization fibre $\nu_Y^{-1}(x)$ has one point for each local branch of the normal-crossing divisor D_Y through x, hence at most three points. For $t \in \nu_a^{-1}(w)$, regard t as a point of the component $E_0 \subset \widetilde{D}_Y$ and set $y_t = \nu_Y(t)$. Since $p(y_t) = p(x)$ and the action on Y is free, there is a unique $\mu_t \in \mathbf{Z}^2$ such that $\Psi_{\mu_t}(y_t) = x$. Set

$$\iota_w(t) = \widetilde{\Psi}_{\mu_t,a}(t) \in \nu_Y^{-1}(x).$$

If $\iota_w(t) = \iota_w(t')$ lies in the component E_ν, then, because $t, t' \in E_0$, one has

$$B_0\mu_t = \nu = B_0\mu_{t'}.$$

Thus $\mu_t = \mu_{t'}$, and applying the inverse lifted action gives $t = t'$. Consequently ι_w is an injection

$$\nu_a^{-1}(w) \hookrightarrow \nu_Y^{-1}(x).$$

Every fibre of ν_a is therefore finite. By the definition of a finite map, ν_a is finite.

Let D_Y^{br} be the complement in D_Y of all pairwise intersections of components. It is a dense open subset. Its quotient $W_a^{\mathrm{br}} = p(D_Y^{\mathrm{br}})$ is open because the quotient map is open, and it is dense because the inverse image of every nonempty open subset of W_a is open and therefore meets D_Y^{br}. By the local normal-crossing equations, $W_a \setminus W_a^{\mathrm{br}}$ is the singular locus of W_a and is a proper closed analytic subset. Every point of D_Y^{br} lies on a unique component. Let

$$\partial E_0 = E_0 \setminus (\mathbf{C}^*)^2$$

be the toric boundary of E_0. The unique component translation to E_0 therefore shows that

$$\nu_a : E_0 \setminus \partial E_0 \longrightarrow W_a^{\mathrm{br}}$$

is bijective. This bijection is locally biholomorphic and hence is a biholomorphism. Therefore ν_a is bimeromorphic.

The surface E_0 is irreducible and normal. Since ν_a is surjective, W_a is irreducible: if $W_a = A \cup B$ were a union of two proper closed analytic subsets, then $E_0 = \nu_a^{-1}(A) \cup \nu_a^{-1}(B)$ would contradict irreducibility of E_0. We have proved that (2.48) is finite, surjective, and bimeromorphic from a normal irreducible space to the reduced irreducible space W_a. The normalization criterion Theorem A.10 therefore identifies ν_a with the normalization of W_a.

Finally, the triangulation has edges. At a point of the toric orbit corresponding to an edge, the local equation of the central fibre is $z_0 z_1 = 0$. The quotient map is locally biholomorphic, so W_a contains a nonempty double-curve stratum with the same local equation. The local normalization has two points over a general point of this stratum, and therefore ν_a is not an isomorphism. Thus W_a is nonnormal. □

On the cusp horodisc, use the adapted fibre coordinates in which Λ_{tor} is represented by the last two columns of the period matrix. Set

$$E_{\exp}(z, \zeta, a) = \big(e^{2\pi i\zeta_1}, e^{2\pi i\zeta_2}, q(z), a\big). \tag{2.49}$$

Translations by Λ_{tor} disappear under the first two exponentials. For $\lambda \in \Lambda$, let $\bar\lambda$ denote its image in $\overline{\Lambda} \cong \mathbf{Z}^2$. The decomposition (2.30) gives

$$E_{\exp}(z, \zeta + \Omega_a(z)\lambda, a) = \Psi_{\bar\lambda}(E_{\exp}(z, \zeta, a)). \tag{2.50}$$

Under the cusp deck transformation the adapted vertical coordinate is unchanged because $R_{g_0} = I_2$, while $q(g_0 z) = q(z)$. Hence $E_{\exp}(g_0 z, \zeta, a) = E_{\exp}(z, \zeta, a)$. Equation (2.31) is the corresponding compatibility of the period lattice with this deck action.

Proposition 2.22. *For the disc $D_0 = D_\varepsilon$ and the smaller horodisc $\mathcal{H}_{0,\varepsilon}$ defined in* (2.36)*, equation* (2.49) *descends to a jointly holomorphic biholomorphism*

$$G_0 : \mathcal{J}_\Delta|_{D_0^*\times\Delta} \xrightarrow{\cong} \mathcal{N}_{0,\Delta}|_{D_0^*\times\Delta}. \tag{2.51}$$

Proof. We first identify the relevant part of the global core. Let

$$\widetilde{\mathcal{J}}^*_{0,\Delta} = \widetilde{\mathcal{J}}_\Delta|_{\mathcal{H}_{0,\varepsilon}\times\Delta}.$$

If $g\widetilde{\mathcal{J}}^*_{0,\Delta} \cap \widetilde{\mathcal{J}}^*_{0,\Delta} \neq \varnothing$, then the projections to U° give

$$g\mathcal{H}_{0,\varepsilon} \cap \mathcal{H}_{0,\varepsilon} \neq \varnothing.$$

The inherited separation property proved immediately after (2.37) therefore implies $g \in \langle g_0\rangle$. By Proposition A.6 and (2.37),

$$\widetilde{\mathcal{J}}^*_{0,\Delta}/\langle g_0\rangle \xrightarrow{\cong} \mathcal{J}_\Delta|_{D_0^*\times\Delta}. \tag{2.52}$$

Thus it remains only to identify the quotient on the left with the punctured toric quotient.

First quotient $\mathbf{C}^2$ by the invariant lattice $\Lambda_{\mathrm{tor}} = \mathbf{Z}\langle\widehat{w}, \widehat{\delta}\rangle$. In the adapted coordinates its two generators are the last two columns of Ω_a, so the map

$$(\zeta_1, \zeta_2) \longmapsto (\mathrm{e}^{2\pi\mathrm{i}\zeta_1}, \mathrm{e}^{2\pi\mathrm{i}\zeta_2})$$

identifies that quotient with $(\mathbf{C}^*)^2$. The remaining quotient lattice is $\overline{\Lambda} \cong \mathbf{Z}^2$, and (2.50) says exactly that its action is (2.41). Thus $E_{\exp}$ induces a bijection after the two lattice quotients are taken.

The cusp deck transformation leaves q and the exponential fibre coordinates unchanged, so taking its quotient replaces the additive coordinate s by the punctured-disc coordinate $q = \mathrm{e}^{2\pi\mathrm{i}s}$. To construct the inverse near a point of the quotient, choose local logarithms of the two nonzero torus coordinates and a local logarithm of q. Dividing the first two logarithms by $2\pi\mathrm{i}$ gives ζ, and dividing the third by $2\pi\mathrm{i}$ gives a lift s of the base point. Changing a fibre logarithm adds a generator of Λ_{tor}; changing the logarithm of q changes s by an integer and therefore changes the chosen point of $\mathcal{H}_{0,\varepsilon}$ by a power of the deck transformation g_0. A different representative under the residual $\mathbf{Z}^2$ action changes ζ by the remaining periods, by (2.50). Hence the local inverse is independent of all choices after passing to the quotients. The local logarithms are holomorphic, so this inverse is holomorphic. Both directions are jointly holomorphic in a. Combining this identification with (2.52) proves (2.51). □

2.5. **Global compactification, choice independence, and smooth type.** Choose the three discs $D_0, D_1, D_2 \subset B$ with pairwise disjoint closures. The four base opens

$$B^\circ, \quad D_0, \quad D_1, \quad D_2$$

cover B, and the only nonempty overlaps involving a filling disc are $D_j^* = D_j \cap B^\circ$.

Glue the disjoint union

$$\mathcal{J}_\Delta \sqcup \mathcal{N}_{0,\Delta} \sqcup \mathcal{N}_{1,\Delta} \sqcup \mathcal{N}_{2,\Delta}$$

by G_0 from (2.51) and by G_j from (2.26). Since the filling discs are disjoint, there are no nontrivial triple overlaps among three different pieces; hence the cocycle condition is automatic once the three displayed maps are fixed.

Lemma 2.23. *The glued topological space $\mathfrak{X}_\Delta$ is Hausdorff and second-countable. The quotient charts determine a unique complex manifold structure on it.*

The filling discs are disjoint, so every gluing class is either a singleton or a two-point class joining the core to one filling. Each piece therefore embeds openly, and disjoint neighbourhoods inside one piece have disjoint open saturations. The complete verification is given in Appendix B.

The local maps glue to a holomorphic map

$$\mathfrak{f} : \mathfrak{X}_\Delta \longrightarrow B \times \Delta. \tag{2.53}$$

Theorem 2.24. *The map (2.53) is proper. The parameter projection*

$$\pi = \mathrm{pr}_\Delta \circ \mathfrak{f} : \mathfrak{X}_\Delta \longrightarrow \Delta$$

is a proper holomorphic submersion. Its fibre over a is the compactification obtained from Ω_a by the two logarithmic fillings and the cusp filling with the chosen valid cusp cutoff. The next proposition constructs canonical biholomorphisms over B showing that the resulting fibre pair is independent of the cutoff and of the parameter disc.

Proof. On the four members of the base open cover, the restrictions of $\mathfrak{f}$ are p°, F_0, F_1, F_2, all of which are proper by Propositions 2.5 and 2.7 and Theorem 2.21. The base $B \times \Delta$ is metrizable. Therefore Lemma A.13 proves that $\mathfrak{f}$ is proper.

Let $K \subset \Delta$ be compact. Since B is compact, $B \times K$ is compact. Hence

$$\pi^{-1}(K) = \mathfrak{f}^{-1}(B \times K)$$

is compact, so π is proper.

On the core, the parameter projection is a submersion by Proposition 2.5. On the finite fillings, it is a submersion by Proposition 2.7; this remains true at the multiple central fibres because the free quotient map is locally biholomorphic and the parameter a is unchanged by the group action. On the cusp filling, it is a submersion by Theorem 2.21; this remains true at the normal-crossing fibre because upstairs the total space is the smooth product $Y_\varepsilon \times \Delta$ and the action is free. These local parameter maps agree under the gluing biholomorphisms, so π is a holomorphic submersion everywhere.

All group actions preserve a, and all gluing maps are defined by formulas whose restriction at a fixed parameter is the corresponding fixed-parameter formula. Thus the fibre over a is the compactification made with the present parameter disc and the present cutoff. Its independence of these choices is established in Proposition 2.25. □

Proposition 2.25. *The relative compactification has the following compatibility properties.*

(i) *Fix a relatively compact parameter disc $\Delta \Subset \mathcal{A}$. If $0 < \varepsilon' < \varepsilon$ are two valid cusp cutoffs for Δ in the sense of Definition 2.10, then the two relative compactifications are canonically biholomorphic over $B \times \Delta$. The biholomorphism is the identity on the core and on the two finite fillings.*

(ii) *Let $\Delta_1, \Delta_2 \Subset \mathcal{A}$ be parameter discs, with any valid cusp cutoffs in the sense of Definition 2.10 chosen in the two constructions. Over $\Delta_1 \cap \Delta_2$, their relative compactifications are canonically biholomorphic. These identifications satisfy the cocycle identity on triple intersections.*

Consequently, for every $a \in \mathcal{A}$, the fibre together with its distinguished map to B is independent, up to the canonical biholomorphisms over B constructed above, of the parameter disc and of the cusp cutoff.

Nested valid cutoffs give invariant open embeddings of cusp models, and the punctured identifications are restrictions of the same exponential formula. Two parameter discs are compared locally using one smaller common cutoff; the restriction identities make these comparisons independent of that cutoff and give the cocycle law. The complete verification is given in Appendix B.

Definition 2.26. For each $a \in \mathcal{A}$, choose once and for all a relatively compact parameter disc $\Delta(a) \Subset \mathcal{A}$ containing a and a valid cusp cutoff for that disc; such choices exist by Lemma 2.11. Denote the resulting relative maps by

$$\mathfrak{f}_{\Delta(a)} : \mathfrak{X}_{\Delta(a)} \longrightarrow B \times \Delta(a), \qquad \pi_{\Delta(a)} = \mathrm{pr}_{\Delta(a)} \circ \mathfrak{f}_{\Delta(a)}.$$

The concrete compact complex manifold and its distinguished map are given by

$$X_a := \pi_{\Delta(a)}^{-1}(a), \qquad f_a := \mathrm{pr}_B \circ \mathfrak{f}_{\Delta(a)}|_{X_a} : X_a \longrightarrow B.$$

For any other parameter disc and valid cutoff containing a, Proposition 2.25 supplies a specified canonical biholomorphism over B from the resulting fibre pair to (X_a, f_a). All subsequent occurrences of X_a and f_a refer to this fixed concrete representative. Since every assertion below is invariant under biholomorphism over B, the initial choice of representative has no effect on any theorem.

Lemma 2.27. *For every $a \in \mathcal{A}$, the manifold X_a is connected and the map $f_a : X_a \to B$ is surjective.*

Proof. Represent (X_a, f_a) by a parameter fibre of one relative compactification, as in Definition 2.26. The fixed-parameter core is connected by Proposition 2.5; the two finite filling slices are connected by Proposition 2.7; and the cusp filling slice is connected by Theorem 2.21. Each filling meets the core over a nonempty punctured annulus. Successively adjoining these three connected fillings therefore gives a connected parameter fibre, and the canonical biholomorphism of pairs makes X_a connected.

The manifold X_a is compact by Theorem 2.24 and Definition 2.26. Hence $f_a(X_a)$ is closed in B. It contains the dense open set B°, because the core map is onto B°. Therefore $f_a(X_a) = B$. □

Corollary 2.28. *For any two admissible parameters $a, a' \in \mathcal{A}$, the fixed-parameter compactifications X_a and $X_{a'}$ are diffeomorphic.*

Proof. More generally, every nonempty finite subset $\{z_1, \ldots, z_N\}$ of the half-plane $\mathcal{A} = \{\Im z < c\}$ lies in a disc $\Delta \Subset \mathcal{A}$. Indeed, write $z_j = x_j + \mathrm{i}(c - \delta_j)$ with $\delta_j > 0$, choose any real x_0, and take

$$R > \max_j \frac{(x_j - x_0)^2 + \delta_j^2}{2\delta_j}.$$

For $w = x_0 + \mathrm{i}(c - R)$ one has $|z_j - w| < R$ for every j; a radius $\max_j |z_j - w| < r < R$ gives the asserted disc.

Apply this observation to a, a'. The projection of the resulting relative compactification is a proper holomorphic submersion by Theorem 2.24; hence its fibres over a and a' are diffeomorphic by Ehresmann's theorem, Theorem A.11. The comparison maps of Proposition 2.25 and Definition 2.26 identify these fibres with the fixed representatives X_a and $X_{a'}$. □

Lemma 2.29. *Let $(X^{\mathrm{A}}, f^{\mathrm{A}})$ be the compact pair of Alpöge* [2, Construction 6.1], *with its specified twists $v_1 = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}$ and $v_2 = -\widehat{\gamma} - 3\widehat{u} + 3\widehat{w}$, and untranslated cusp gluing. For its chosen admissible period function β^{A}, there is a unique constant $a_{\mathrm{A}} \in \mathcal{A}$ for which $\beta^{\mathrm{A}} = \beta_0 + a_{\mathrm{A}}$. For this parameter there is an isomorphism of pairs*

$$(X^{\mathrm{A}}, f^{\mathrm{A}}) \cong (X_{a_{\mathrm{A}}}, f_{a_{\mathrm{A}}}) \quad \text{over } B.$$

Proof. Use the same oriented $(3, 4, \infty)$ triangle and the same affine normalization of its quotient: the order-three, order-four, and cusp points are $0, 1, \infty$, respectively. This identifies the marked coverings and their generators g_1, g_2, g_0. The integral matrices (2.2) are Alpöge's dual monodromy matrices in the ordered basis $(\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$. His τ and μ have the generator laws (2.7) and (2.8), and the same cusp normalization as ours. The uniqueness assertions in [2, Theorem 3.4(i),(ii)] identify them. His β^{A} and our β_0 satisfy (2.9), and both $\beta + \tau$ extend across the cusp. Their difference descends to a holomorphic function on $B = \mathbf{P}^1$, hence is a constant a_{A}:

$$\beta^{\mathrm{A}} = \beta_0 + a_{\mathrm{A}}, \qquad \Pi^{\mathrm{A}} = \Omega_{a_{\mathrm{A}}}.$$

This also fixes the vertical matrices R_g, since they are determined by the same period matrix and integral monodromy.

Alpöge's admissibility condition is $\mathscr{D}_{a_{\mathrm{A}}}(z) < 0$ for every $z \in U$. By Proposition 2.2, this invariant function is continuous on $B \setminus \{p_0\}$ and tends to $-\infty$ at p_0; it therefore attains a strictly negative maximum. Since $\mathscr{D}_{a_{\mathrm{A}}} = \mathscr{D}_0 + \Im a_{\mathrm{A}}$, we obtain $M_0 + \Im a_{\mathrm{A}} < 0$, or $a_{\mathrm{A}} \in \mathcal{A}$.

The four local pieces now agree at a_{A}. On the smooth core they are quotients of the same marked period family by the same deck action. At p_j, the vectors in (2.19) equal the two source vectors; hence the finite actions (2.22) and the logarithmic translations (2.25) agree with Alpöge's finite fillings and gluing maps. At the cusp, both constructions use the triangles $\{v, v + e_1, v + e_2\}$ and $\{v + e_1, v + e_2, v + e_1 + e_2\}$ for $v \in \mathbf{Z}^2$, the same $B_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and the same decomposition $Z_{a_{\mathrm{A}}} = sB_0 + C_{a_{\mathrm{A}}}(q)$. Thus the toric lattice action (2.40) and the exponential gluing (2.49) agree. In particular, neither construction inserts a fibre translation at the cusp. In the discrete notation of [2, Section 6.4], these facts say $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$, with the last two values given by (2.21).

Take the finite linearizing discs and the cusp cutoff to be common smaller choices in the two constructions. On these choices, the identity on each of the four pieces respects every gluing map, hence descends to a biholomorphism of pairs over B. Restriction to smaller discs removes only punctured collars already covered by the core. Changing a finite logarithm branch changes its fibre translation by an integral period; changing the cusp logarithm changes C by an integral multiple of B_0, whose exponential action is unchanged. Thus the comparison is independent of these choices, consistently with [2, Proposition 6.3]. The intrinsic representative X_{a_A} is independent of its parameter disc and cusp cutoff by Proposition 2.25 and Definition 2.26, giving the stated isomorphism. □

Corollary 2.30. *Every compact threefold X_a in the admissible family is diffeomorphic to S^6.*

Proof. By Lemma 2.29, X_{a_A} is Alpöge's threefold, which is diffeomorphic to S^6 by [2, Theorem 8.1]. The smooth type is constant throughout $\mathcal{A}$ by Corollary 2.28. □

3. Cyclic coverings and Oka maps

3.1. **Oka criteria and cyclic period coverings.** An open subset is called analytic-Zariski open if its complement is a closed analytic subset of the stated ambient complex space. We recall the Oka criteria used in the argument.

Definition 3.1. A holomorphic submersion $p : Y \to S$ satisfies POPAI if the following holds. Let T be a reduced Stein space, $T' \subset T$ a closed complex subvariety, $K \subset T$ a compact $\mathcal{O}(T)$-convex set, and $P_0 \subset P$ compact Hausdorff parameter spaces. Given a continuous family of holomorphic maps $F : P \times T \to S$ and a continuous lifting $f_0 : P \times T \to Y$, assume that there is a single open neighbourhood $U \supset K$ in T for which

$$f_0|_{P_0 \times T}, \qquad f_0|_{P \times T'}, \qquad f_0|_{P \times U}$$

are families with holomorphic slices. In particular, U is independent of the parameter in P; separate parameter-dependent neighbourhoods do not suffice. There is then a continuous homotopy of liftings f_t, $t \in [0, 1]$, over the fixed F, ending in a family holomorphic on T. The homotopy is fixed on $(P_0 \times T) \cup (P \times T')$. For each t, the slices of f_t are holomorphic on an open neighbourhood $U_t \supset K$ common to all parameters in P, and the approximation to f_0 on $P \times K$ is as close as prescribed, uniformly for $t \in [0, 1]$. The neighbourhood quantifiers follow [24, Definition 1.1]; arbitrary compact Hausdorff parameter spaces are covered by [24, Corollary 5.7]. An *Oka map* is such a submersion that is also a Serre fibration. A complex manifold is Oka when its map to a point is an Oka map.

Theorem 3.2. *A holomorphic submersion $p : Y \to S$ satisfies POPAI if and only if it is convexly elliptic: there is an open cover $\{V_\alpha\}$ of S such that, for every compact convex $K \subset \mathbf{C}^n$ and holomorphic $f : \mathrm{Op}\, K \to Y$ with $f(K) \subset p^{-1}(V_\alpha)$ for some α, there is a holomorphic spray $s : \mathrm{Op}\, K \times \mathbf{C}^N \to Y$ with*

$$s(z, 0) = f(z), \qquad p(s(z, w)) = p(f(z)), \qquad d_w s(z, 0)(\mathbf{C}^N) = \ker dp_{f(z)} \quad (z \in K).$$

If every point of Y has an analytic-Zariski open neighbourhood on which p satisfies POPAI, then p satisfies POPAI.

These are [24, Theorem 1.3 and Corollary 5.13]. For a map to a point, they include analytic-Zariski localization of the Oka property. We also use covering invariance: a complex manifold is Oka if and only if any surjective unramified holomorphic covering of it is Oka [20, Proposition 5.6.3].

For the relative family in Theorem 2.24, set

$$\mathcal{S}_j = (\mathfrak{f}_\Delta^{-1}(\{p_j\} \times \Delta))_{\mathrm{red}}, \qquad \mathcal{U}_j = \mathfrak{X}_\Delta \setminus \mathcal{S}_j.$$

The closed analytic sets $\mathcal{S}_1, \mathcal{S}_2$ are disjoint. Their complements form an analytic-Zariski cover of the total space. Throughout this section, $\pi = \pi_\Delta$ and $a_* \in \Delta$.

We first normalize the finite base changes.

The normalized base maps from $\mathbf{C}_b$ are

$$\rho_4(b) = \frac{1}{1 - b^4} : \mathbf{C}_b \longrightarrow \mathbf{P}^1 \setminus \{0\}, \qquad \rho_3(b) = \frac{b^3}{b^3 - 1} : \mathbf{C}_b \longrightarrow \mathbf{P}^1 \setminus \{1\}.$$

Their poles are interpreted as the value ∞ in $\mathbf{P}^1$. The first has degree four and is ramified only over 1, while the second has degree three and is ramified only over 0. They are unramified over the cusp; its inverse images are respectively $b^4 = 1$ and $b^3 = 1$.

For $(m, j) = (4, 1)$ or $(3, 2)$, let $\mathcal{M}_m$ be the normalization of

$$\mathcal{U}_j \times_{(\mathbf{P}^1 \setminus \{p_j\}) \times \Delta} (\mathbf{C}_b \times \Delta).$$

There are maps

$$r_m : \mathcal{M}_m \longrightarrow \mathcal{U}_j, \qquad h_m : \mathcal{M}_m \longrightarrow \mathbf{C}_b \times \Delta. \tag{3.1}$$

The map r_m is a finite unramified holomorphic covering of degree m. Indeed, away from the remaining multiple fibre this is the ordinary unramified base change. At that fibre the local model in Section 2 is

$$\mathcal{P}_{k,\Delta}/\langle \widetilde{g}_k \rangle, \qquad t = s^m,$$

with a free affine cyclic action of order m. Normalizing the root base change $t = s^m$ recovers $\mathcal{P}_{k,\Delta}$, and the map back to the quotient is unramified because the action is free. The local identification can differ by a root of a nowhere-zero base function; this is independent of a. All these constructions are relative to Δ. In particular, $\mathcal{M}_m$ is smooth.

Their fibres at a fixed parameter are the finite covers used in the geometric realization of Theorem 3.11. The normalized bases carry the required cyclic coverings.

Lemma 3.3. *For each $m = 3, 4$ there is a connected regular unramified holomorphic covering*

$$q_m : Q_m \longrightarrow \mathcal{M}_m$$

with deck group $\mathbf{Z}$, whose restriction to a smooth torus fibre is the cover associated to $\Lambda_0 \subset \Lambda$. There is a biholomorphism

$$Q_m \;\cong\; Q_{m,a_*} \times \Delta \tag{3.2}$$

over $\mathbf{C}_b \times \Delta$, where $Q_{m,a_*} = Q_m|_{a_*}$. No assertion that this covering is the universal covering is needed.

Proof. We distinguish the existence of the covering from its holomorphic triviality.

Choose a locally finite good cover $\{V_\alpha\}$ of $\mathbf{C}_b$ by simply connected open sets. Each cusp value and the point $b = 0$ lies in a single designated member, and every nonempty overlap avoids these special values. We may take all nonempty finite intersections to be contractible.

Over an ordinary member, or a member about $b = 0$ after the normalized root base change, replace the full period lattice by Λ_0. The resulting cover is

$$Q_\alpha \times \Delta, \qquad Q_\alpha = (V_\alpha \times \mathbf{C}^2)/\Omega_{a_*}\Lambda_0.$$

Here a local lift of the b-chart to the uniformizing z-coordinate is understood, also on a normalized finite-root chart. By (2.12), the periods of Λ_0 agree for a and a_* on this lift. Thus the displayed quotient is the same for every a; changes of lift preserve this sublattice. It is a regular cyclic cover of the local full-lattice quotient.

At a cusp, use the notation of Section 2:

$$Z_a = sB_0 + C_a(q), \quad C_a(q) = C_0(q) + aE_{21}, \quad \Psi_{\lambda,a}(y) = \exp(2\pi i C_a(q(y))\lambda) \cdot \Phi_\lambda(y).$$

The local total space is $(Y_\varepsilon \times \Delta)/\mathbf{Z}^2$, with $\lambda = (\lambda_\gamma, \lambda_u)$. Since $E_{21}(0, n)^t = 0$, the subgroup action $\Psi_{(0,n),a}$ is independent of a. Hence the required local cyclic cover is

$$(Y_\varepsilon/\langle \Psi_{(0,1)} \rangle) \times \Delta \longrightarrow (Y_\varepsilon \times \Delta)/\mathbf{Z}^2. \tag{3.3}$$

Freeness and proper discontinuity follow by restriction from the relative cusp action already proved in Section 2. On its punctured part, (3.3) is exactly the cover for Λ_0: the periods $\widehat{w}, \widehat{\delta}$ have been removed by exponentiation and $\widehat{u}$ is still quotiented out. Transport to the other cusp lifts preserves γ.

There are three kinds of nonempty overlap: ordinary–ordinary, ordinary–finite-root, and ordinary–cusp. The designated special members are disjoint, so no overlap contains two special values. On each overlap the base is contractible and all fibres are smooth tori. In the first case the transition is a period/marking change; in the second it is the normalized finite-root change followed by the affine translation; in the third it is the punctured cusp exponential gluing. Each preserves $\Lambda_0 = \ker \gamma$ and γ itself. Thus each transition lifts to a map

$$F_{\alpha\beta,a} : Q_\beta|_{V_\alpha \cap V_\beta} \longrightarrow Q_\alpha|_{V_\alpha \cap V_\beta}.$$

Fix a lift at one point of the contractible overlap times Δ. The smooth torus bundle there has fundamental group identified with its fibre lattice, and both covers correspond to Λ_0. The lifting criterion extends this choice over the product. Since the covering maps are local biholomorphisms, the resulting lift is holomorphic in both b and a. A different choice changes it by a constant integer deck translation. On triple overlaps their failure to satisfy the cocycle identity is thus an integer-valued Cech 2-cocycle $n_{\alpha\beta\gamma}$. This integer is locally constant in both b and a, and hence independent of a. Conjugation by a transition preserves the positive generator: the ordinary generator is the class with $\gamma = 1$, its normalized-root transport has the same γ-value, and on a cusp chart it is $\Psi_{(1,0),a}$. Thus the coefficient sheaf is the constant sheaf $\mathbf{Z}$.

The good cover computes the cohomology of the constant sheaf $\mathbf{Z}$. Since $\mathbf{C}$ is contractible, $H^2(\mathbf{C}, \mathbf{Z}) = 0$. Adjusting the lifts by an integer-valued 1-cochain kills this 2-cocycle. The correcting deck maps are jointly holomorphic in a, so the corrected transitions satisfy the cocycle identity for all a. They glue the local covers to $q_m : Q_m \to \mathcal{M}_m$. Since they glue actual local covering maps over the already Hausdorff manifold $\mathcal{M}_m$, the result is a Hausdorff unramified holomorphic covering. Each fibre over $\mathbf{C}_b$ is connected: on ordinary and normalized finite charts it is $\mathbf{C}^2/\Omega_{a_*}\Lambda_0$, and at a cusp it is the quotient of the connected central toric divisor by $\langle \Psi_{(0,1)} \rangle$. The local deck group is the marked quotient $\Lambda/\Lambda_0 \simeq \mathbf{Z}$, with positive generator preserved by every transition, so these deck maps glue to the global deck group $\mathbf{Z}$.

The action Θ extends over the finite fillings because $R_g e_2 = e_2$ and all added terms are translations. At the cusp it is multiplication of the second vertical torus coordinate. It preserves q, extends over the toric boundary, and commutes with the shear and lattice actions by Proposition 2.12. Every lifted transition is therefore $\mathbf{C}^*$-equivariant for Θ. More precisely, on each overlap there is $h_{\alpha\beta} \in \mathcal{O}(V_\alpha \cap V_\beta)$ such that

$$F_{\alpha\beta,a} = \Theta_{\exp(2\pi i (a-a_*) h_{\alpha\beta}(b))} \circ F_{\alpha\beta,a_*}. \tag{3.4}$$

Here is a direct check from the construction formulas. A period translation by λ changes, between a_* and a, by $(a-a_*)\gamma(\lambda)e_2$. A finite affine translation by v_k/m changes by $(a-a_*)\gamma(v_k)e_2/m$. A logarithmic filling translation satisfies

$$\sigma_{k,a} - \sigma_{k,a_*} = \frac{(a-a_*)\gamma(v_k)}{2\pi i} \log s_k \, e_2. \tag{3.5}$$

A single-valued branch is chosen on each contractible overlap. Changes of branch add an integer multiple of the already included period translation. The linear marking maps and exponential cusp gluing are independent of a and preserve the action Θ. Finally, the cusp deck maps themselves have parameter difference $\Theta_{\exp(2\pi i (a-a_*)\lambda_\gamma)}$. These statements cover the three overlap types listed above and are preserved under composition and inversion, since all maps preserve the base and commute with Θ. They prove (3.4), including the integer deck corrections used in the preceding paragraph. More explicitly, let $D_{\alpha,a}$ be the positive local deck generator: it is the $\widehat{\gamma}$ period translation on an ordinary chart and $\Psi_{(1,0),a}$ on a cusp chart. Then, for every $n \in \mathbf{Z}$,

$$D^n_{\alpha,a} = \Theta_{\exp(2\pi i n (a-a_*))} \circ D^n_{\alpha,a_*}.$$

An integer correction therefore only adds an integer to $h_{\alpha\beta}$. Its coefficient remains a function of the base alone.

The corrected transitions and their specializations at a_* satisfy the cocycle identity. Substituting (3.4) gives

$$\Theta_{\exp(2\pi i (a-a_*)(h_{\alpha\beta}+h_{\beta\gamma}-h_{\alpha\gamma}))} = \mathrm{id}.$$

On a smooth fibre the action Θ is effective. Indeed, if ue_2 is a period in Λ_0, its first coordinate gives $n\tau + k = 0$ for integers n, k; as $\Im\tau > 0$, this forces $n = k = 0$, and then $u \in \mathbf{Z}$. Letting a vary in an open disc now shows

$$h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}.$$

Cartan's Theorem B gives $H^1(\mathbf{C}, \mathcal{O}) = 0$, so

$$h_{\alpha\beta} = b_\alpha - b_\beta, \qquad b_\alpha \in \mathcal{O}(V_\alpha).$$

Change the αth local trivialization by

$$A_{\alpha,a} = \Theta_{\exp(-2\pi i (a-a_*) b_\alpha(b))}.$$

Equivariance and the last identity imply

$$A_{\alpha,a} \circ F_{\alpha\beta,a} \circ A^{-1}_{\beta,a} = F_{\alpha\beta,a_*}.$$

The new transitions are therefore independent of a, proving (3.2). The multipliers are nowhere zero and holomorphic on the full base neighbourhoods. Because Θ extends over the toric boundary, this change of trivialization is valid at the central strata as well, not just on the punctured part. □

Equation (3.2) trivializes the cyclic covering family. Its deck maps retain their dependence on a, and the Oka argument below uses the bands assembled from these local models.

3.2. **Conic-band geometry on the cyclic coverings.** The cyclic covers of Section 3 carry a conic-band geometry. We specify its exact requirements, construct them on every admissible member, and then prove the Oka property of the two open complements.

Definition 3.4. For fixed $a_* \in \mathcal{A}$, write $U_j = X_{a_*} \setminus S_{j,a_*}$. The conic-band package consists of the following data for $(m, j) = (4, 1)$ and $(3, 2)$.

(G1) There is a finite unramified holomorphic covering $N_m \to U_j$, of degree m, and a surjective unramified holomorphic covering $\widehat{N}_m \to N_m$. The space $\widehat{N}_m$ is covered by analytic-Zariski open subsets isomorphic to a fixed smooth threefold Y_m. The realization below identifies them with the finite and cyclic covers of Section 3.

(G2) There is a smooth algebraic surface E_m, covered by two algebraic Zariski open charts with the following models and regular base functions:

$$W_3 = \{ab = c(c + 2i)\}, \qquad z = (4a^2 - b)/3, \tag{3.6}$$

$$W_4 = \{ab = c(4c^2 - 3)\}, \qquad z = (a - b)/(2i). \tag{3.7}$$

There is a holomorphic line bundle $L_m \to E_m$. Set

$$\Delta_3(z) = z^3 - 1, \qquad \Delta_4(z) = 1 - z^4, \qquad H_m = \{A \otimes B = \Delta_m(z)\} \subset L_m \oplus L_m^{-1}.$$

A proper small resolution $\rho_m : Y_m \to H_m$ has exceptional locus a finite disjoint union of closed analytic curves $C_i \simeq \mathbf{P}^1$ and restricts to a biholomorphism

$$U_m^{\mathrm{reg}} := Y_m \setminus \bigcup_i C_i \xrightarrow{\sim} H_m^{\mathrm{reg}}.$$

(G3) Set $Y_{m,i} = Y_m \setminus \bigcup_{k \neq i} C_k$. There is a normal finite Galois double cover $\bar{p}_i : \bar{Y}_{m,i} \to Y_{m,i}$ with group $G_i = \{1, \tau_i\}$. It is unramified outside a closed analytic divisor $D_{m,i}$ disjoint from C_i. A G_i-invariant open submanifold $\bar{N}_i \subset \bar{Y}_{m,i}$ contains $\bar{p}_i^{-1}(C_i)$, is contained in $\bar{p}_i^{-1}(Y_{m,i} \setminus D_{m,i})$, and has closed analytic complement in $\bar{Y}_{m,i}$.

(G4) The open manifold $\bar{N}_i$ has a cover by four analytic-Zariski open incidence charts, consisting of two selected charts and their sign transforms. Each order-three chart is precisely a complement

$$(\mathbf{C}_w^* \times \mathbf{C}_A \times \mathbf{C}_p) \setminus \bigcup_{k=1}^{s} \{w = w_k, A = 0\}, \qquad w_k \neq 0. \tag{3.8}$$

Each order-four chart is precisely a complement

$$Z(L, q) \setminus \bigcup_{k=1}^{s} \ell_k(L), \tag{3.9}$$

where $T = (\mathbf{C}^*)^2_{\lambda,t}$, $L \to T$ is a holomorphic line bundle, $q \in \mathcal{O}(\mathbf{C}^*)$, and

$$Z(L, q) = \{A \in L_{(\lambda,t)},\ p \in L^\vee_{(\lambda,t)} : \langle A, p\rangle = t - q(\lambda)\},$$
$$\ell_k(L) = \{\lambda = \lambda_k,\ t = q(\lambda_k),\ A = 0\}, \qquad \lambda_k \neq 0, \quad q(\lambda_k) \neq 0.$$

The coordinate p is unrestricted on each deleted line. In the order-four source charts one has $q(\lambda) = i\zeta - 2\lambda^3 - 3\zeta^2\lambda$, with $\zeta^4 = 1$; the argument permits any displayed q. The sign transforms have the same intrinsic form. No trivialization of L on T is assumed.

The word precisely in the last condition means that the chart parameters range over the full displayed models and that every deleted p-line is complete. Invariance and the closed analytic complement in (G3) are used for finite-quotient descent. The following proof constructs these objects on the actual cyclic covers, including their global transitions.

The geometric package above uses the cyclic coverings already constructed in Section 3. The local coordinate z in this subsection is the coordinate b of Equation (3.1); it is renamed to agree with the equations of the elliptic surfaces below.

Lemma 3.5. *For $m = 3, 4$, the cyclic covering Q_{m,a_*} of Lemma 3.3 has an analytic-Zariski open subset Y_m such that its deck translates cover Q_{m,a_*}. There is a smooth analytic surface $\mathcal{E}_m^{\mathrm{an}}$ proper over $\mathbf{C}_z$, with* *precisely the m nodal fibres $z^m = 1$, and a holomorphic line bundle $L_m \to \mathcal{E}_m^{\mathrm{an}}$ for which Y_m is a proper small resolution of*

$$H_m = \{A \otimes B = \Delta_m(z)\} \subset L_m \oplus L_m^{-1}, \qquad \Delta_3 = z^3 - 1, \quad \Delta_4 = 1 - z^4.$$

Its exceptional locus consists of one complete $\mathbf{P}^1$ over the node of each nodal fibre.

Proof. The normalized root maps $\rho_3(z) = z^3/(z^3 - 1)$ and $\rho_4(z) = 1/(1 - z^4)$ give the finite unramified covers $N_m = M_{m,a_*} \to U_j$ of degree m, where $(m, j) = (3, 2), (4, 1)$; see Equation (3.1). The further cover $Q_{m,a_*} \to N_m$ retains precisely

$$\Lambda_0 = \ker \gamma = \mathbf{Z}\langle \widehat{u}, \widehat{w}, \widehat{\delta} \rangle.$$

Over an ordinary base chart its fibre is

$$\mathbf{C}^2/\mathbf{Z}\langle (\tau, \mu), (1, 0), (0, 1) \rangle. \tag{3.10}$$

At a cusp $z = \zeta$, $\zeta^m = 1$, the local model is $Y_\varepsilon/\langle \Psi_{(0,1)} \rangle$, as in Equation (3.3). Write its torus coordinates as (x_1, x_2, q). Since $B_0(0, 1)^t = (1, 0)^t$, the generator acts by

$$(x_1, x_2, q) \longmapsto \big(q\alpha(q)x_1, \beta(q)x_2, q\big), \tag{3.11}$$

where α and β are holomorphic units on the full cusp disc. The other deck generator has $B_0(1, 0)^t = (0, -1)^t$, and hence shifts the second, or row, index by one.

The fan of Proposition 2.9 has vertices $(i, j, 1)$. Projection

$$(i, j, r) \longmapsto (i, r) \tag{3.12}$$

takes its cones into those of the one-dimensional Tate fan. On the chart between columns i and $i + 1$, the characters

$$u = x_1 q^{-i}, \qquad v = x_1^{-1} q^{i+1}, \qquad uv = q \tag{3.13}$$

are holomorphic. The action in (3.11) identifies adjacent columns by $(u, v) \mapsto (\alpha u, \alpha^{-1} v)$. Its quotient is the usual one-node Tate family: the column components of the central chain become one irreducible nodal rational fibre. It is proper over a smaller closed cusp disc. Indeed, for $q \neq 0$ every orbit has a representative in a closed annular fundamental region for multiplication by $q\alpha(q)$, and the toric boundary supplies its compact limit at $q = 0$. The quotient is locally $uv = q$, so its total space is smooth.

On the ordinary charts the free x_2-scaling quotient of (3.10) is the elliptic curve $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$. The transitions used to construct Q_{m,a_*} commute with this scaling and their first coordinate is independent of x_2; the cusp transitions induce (3.13). They therefore glue the ordinary quotients and the nodal cusp quotients to a smooth surface $p_m : \mathcal{E}_m^{\mathrm{an}} \to \mathbf{C}_z$. Properness is local on the target and follows on the ordinary and cusp charts as just checked. Its fibres are connected; the only singular fibres occur at $z^m = 1$, each has one node and multiplicity one. In particular the fibration is relatively minimal.

At each cusp retain the central toric divisors in rows $j = 0, 1$ and delete those in every other row. Retain all noncentral points, and call the resulting subset of Q_{m,a_*} Y_m. The deleted divisors form a locally finite analytic family before the column quotient. The quotient action only translates columns, so they remain locally finite after quotient; there are only finitely many cusps. Their union is consequently closed analytic. Every simplex of the fan has vertices in at most two consecutive rows. Thus every central point lies in a deck translate of Y_m, and

$$Q_{m,a_*} = \bigcup_{n \in \mathbf{Z}} D^n Y_m. \tag{3.14}$$

Each translate is analytic-Zariski open and biholomorphic to Y_m.

On the two retained rows the characters

$$A = x_2, \qquad B = q/x_2 \tag{3.15}$$

have nonnegative pairing with every retained ray and hence extend holomorphically. On the chart (3.13), their relation is $AB = q = uv$. The two triangles in each unit square give the standard small resolution of this conifold. Explicitly, its two affine charts are

$$v = Ap, \quad B = up, \qquad u = Br, \quad A = vr, \qquad r = 1/p \tag{3.16}$$

on their common locus. The inverse image of the origin is the complete $\mathbf{P}^1$ obtained from the two affine exceptional lines; the incidence realization in $H \times \mathbf{P}^1$ makes the contraction proper. Away from the origin these charts give the identity on $AB = uv$.

Over $\Delta_m(z) \neq 0$, the x_2-scaling is free and defines a principal $\mathbf{C}^*$-bundle, hence a holomorphic line bundle. Over a nodal fibre the scaling can have fixed points: in the local equation $AB = uv$, the point $A = B = 0$ over a smooth point of $uv = 0$ is fixed. The equivariant transition functions extend the line bundle across these fibres. At a node, (3.11) multiplies A by the unit $\beta(q)$ and B by its inverse. The ordinary-to-cusp transitions are scaling-equivariant; on the punctured overlap they multiply A by a holomorphic unit pulled back from the elliptic base. These units satisfy the cocycle identity inherited from the transitions of Q_{m,a_*}. Hence L_m extends over all of $\mathcal{E}_m^{\mathrm{an}}$, and B is a coordinate in its dual. At $z = \zeta$ the local cusp parameter q and $\Delta_m(z)$ have the same simple zero. Multiplying the local B-coordinate by the unit Δ_m/q changes $AB = q$ into $A \otimes B = \Delta_m$, compatibly with the dual transitions. Thus the local contractions glue to $Y_m \to H_m$. They are isomorphisms off the m node points and proper near each node by (3.16); elsewhere they are isomorphisms. Properness is local on H_m, proving the assertion. ◻

Lemma 3.6. *There are fibre-preserving biholomorphisms $\mathcal{E}_m^{\mathrm{an}} \simeq E_m^{\mathrm{an}}$, where E_m is the Weierstrass surface over $\mathbf{A}_z^1$*

$$E_3 : \ y^2 = 4x^3 - 3zx - 1, \qquad E_4 : \ y^2 = 4x^3 - 3x - z^2, \tag{3.17}$$

with its section at infinity included. Each E_m is covered by two algebraic Zariski open subsets with the models (3.6) *and* (3.7)*. Under this identification the nodes are*

$$e_\zeta = (-\zeta^2/2, 0, \zeta), \qquad \zeta^m = 1. \tag{3.18}$$

Proof. The surface $\mathcal{E}_m^{\mathrm{an}}$ is a relatively minimal elliptic fibration without multiple fibres over the noncompact Riemann surface $\mathbf{C}$. Buzzard and Lu's [19, Lemma 3.8] section theorem therefore gives a holomorphic section O. The section can be chosen to avoid the finitely many nodes by the same lemma. Translation by a section through the smooth group part extends across an I_1 node: in a Tate chart it is multiplication by a nonzero holomorphic parameter.

The degree-three line bundle $\mathcal{O}(3O)$ is very ample on each smooth or irreducible nodal genus-one fibre. Indeed, for every length-two subscheme Z of such a fibre C, duality on the Gorenstein curve gives $H^1(C, \mathcal{O}_C(3O) \otimes \mathcal{I}_Z) \cong \mathrm{Hom}(\mathcal{O}_C(3O) \otimes \mathcal{I}_Z, \omega_C)^* = 0$: a nonzero map from the rank-one torsion-free sheaf of degree 1 to $\omega_C \simeq \mathcal{O}_C$ would be injective and contradict the degree inequality. Thus global sections separate every length-two subscheme. Fibrewise Riemann–Roch and the vanishing of H^1 in positive degree give the required base change for $p_{m*}\mathcal{O}(kO)$. Holomorphic line bundles on $\mathbf{C}$ are trivial, and their extension sequences split by Cartan's Theorem B. Global functions with pole orders two and three along O therefore yield, after completing the square and cube, an equation

$$y^2 = 4x^3 - g_2(z)x - g_3(z), \qquad g_2, g_3 \in \mathcal{O}(\mathbf{C}). \tag{3.19}$$

The relative cubic embeds each fibre, including each irreducible nodal fibre, so this equation describes the original surface rather than merely a birational model.

The marked periods give

$$j_3(z) = 1728\,\frac{z^3}{z^3 - 1}, \qquad j_4(z) = \frac{1728}{1 - z^4}. \tag{3.20}$$

For $m = 3$, g_3 cannot vanish at a smooth fibre, since j_3 there never equals 1728; at a nodal fibre both g_2 and g_3 are nonzero. It is therefore a holomorphic unit. Its sixth root exists on simply connected $\mathbf{C}$, and scaling (x, y) makes $g_3 = 1$. The formula

$$j = 1728\,\frac{g_2^3}{g_2^3 - 27g_3^2} \tag{3.21}$$

then gives $g_2^3 = 27z^3$; a remaining constant root-of-unity scaling makes $g_2 = 3z$. For $m = 4$, j_4 never vanishes at a smooth fibre, so g_2 is a unit. A fourth-root scaling makes $g_2 = 3$, after which (3.21) gives $g_3^2 = z^4$. The remaining constant choice makes $g_3 = z^2$. This proves (3.17). Its discriminants are $27(z^3 - 1)$ and $27(1 - z^4)$; the equations and their x-derivatives give (3.18). The total-space gradients are nonzero at these points.

The complement of $O = [0 : 1 : 0]$ in E_3 has coordinates

$$a = x, \quad c = y - i, \quad b = 4x^2 - 3z, \qquad ab = c(c + 2i), \quad z = (4a^2 - b)/3. \tag{3.22}$$

For $E_4 \setminus O$, the coordinates

$$a = y + iz, \quad b = y - iz, \quad c = x, \qquad ab = c(4c^2 - 3), \quad z = (a - b)/(2i) \tag{3.23}$$

give the second model. The algebraic sections $P_3 = (0, i)$ and $P_4 = (0, iz)$ are disjoint from O and pass through the smooth part of every nodal fibre, since $-\zeta^2/2 \neq 0$. Translation by P_m on the smooth group part extends over a nodal fibre in its multiplicative Tate coordinate. It is an algebraic rational map with no indeterminacy after that extension, and its inverse is translation by $-P_m$; hence it is an algebraic automorphism over $\mathbf{A}^1_z$. The translates $E_m \setminus P_m$ of the two charts above are the required second Danielewski charts. As $O \cap P_m$ is empty, the two charts cover E_m. □

3.3. **Double covers and complete incidence charts.** The following local calculation records the global extent of an incidence chart. It distinguishes a complete exceptional affine line from a small coordinate neighbourhood of one of its points.

Lemma 3.7. *Let T be an open subset of a smooth surface, let $L \to T$ be a line bundle, and suppose that $H = \{A \otimes B = fg\} \subset L \oplus L^{-1}$. At every node under consideration assume $A = B = f = g = 0$ and df, dg linearly independent. Remove from the target any other nodes whose exceptional curves have already been discarded. The full incidence chart*

$$Ap = f, \qquad B = pg, \qquad p \in L^{-1} \tag{3.24}$$

embeds in a small resolution of the remaining nodes after deletion, at each node with the opposite ruling, of that node's entire affine p-line. No part of a retained exceptional affine line is deleted. If T is analytic-Zariski open in a global base and the relevant divisor components are locally finite, the image is analytic-Zariski open in the global small resolution.

Proof. The chart in (3.24) is one affine chart of the incidence resolution $\mathrm{Bl}_{(A,f)} H$. Its complementary chart has $Br = g$, $A = rf$, with reciprocal exceptional coordinates $r = 1/p$ after the unit in the factorization is included. In the same ruling, the displayed chart contains the whole exceptional affine p-line and omits only the infinity divisor.

For the opposite ruling, use its two charts

$$A = gs, \quad f = Bs, \qquad \text{and} \qquad g = Ar, \quad B = fr. \tag{3.25}$$

Away from the exceptional curve, the identity on the regular part of H identifies (3.24) with the locus $g \neq 0$ in the first chart and $A \neq 0$ in the second. Its complement is the strict transform of the appropriate component of $\{A = 0\}$, including the complete exceptional curve over the opposite-ruling node. The identity is a bijective local biholomorphism on the retained locus, hence an open immersion. The omitted infinity and strict-transform divisors are closed analytic. The same remains true globally after adding the inverse image of the analytic complement of T; local finiteness prevents accumulation of distinct divisor components in the target. □

Lemma 3.8. *For every exceptional curve C_i of Y_m, there is a connected normal finite Galois double cover $\bar{p}_i : \bar{Y}_{m,i} \to Y_{m,i}$ as in* (G3). *Its branch divisor is disjoint from C_i, and $\bar{p}_i^{-1}(C_i)$ is the disjoint union of two complete copies of $\mathbf{P}^1$.*

Proof. Let e_ζ be the image of C_i in E_m. On the Weierstrass surface, x has a double pole along the section O. Consequently $x - \zeta^2$ is a global section of $\mathcal{O}_{E_m}(2O)$. Taking its square root in $\mathcal{O}_{E_m}(O)$ and normalizing produces a finite double cover $\widetilde{E}_{m,\zeta} \to E_m$, written on the affine part as

$$\lambda^2 = x - \zeta^2. \tag{3.26}$$

This is a line-bundle construction also over O: the double pole creates no extra branch component. On a general smooth elliptic fibre the divisor of $x - \zeta^2$ has two distinct simple zeros, so its meromorphic function field extension is nontrivial and the cover is connected.

The normalization of $Y_{m,i} \times_{E_m} \widetilde{E}_{m,\zeta}$, denoted $\bar{Y}_{m,i}$, is finite and normal. The sign involution $\lambda \mapsto -\lambda$ gives its Galois group. The normalization is connected: over the complement of the nodal and branch fibres, $Y_{m,i} \to E_m$ has connected $\mathbf{C}^*$ fibres, and the pullback of this bundle to the connected double cover of that dense base is connected. Its closure is all of $\bar{Y}_{m,i}$. The cover is unramified away from the zero divisor $D_{m,i}$ of $x - \zeta^2$, pulled back to $Y_{m,i}$. At the node,

$$x(e_\zeta) - \zeta^2 = -\frac{3}{2}\zeta^2 \neq 0. \tag{3.27}$$

The function is constant with this value on C_i, so $D_{m,i} \cap C_i = \emptyset$. Since $C_i \simeq \mathbf{P}^1$ is simply connected, the unramified double cover above it splits into two complete copies of $\mathbf{P}^1$. Their λ-values are $\lambda_\pm$, where

$$\lambda_\pm^2 = -\frac{3}{2}\zeta^2, \qquad \lambda_- = -\lambda_+. \tag{3.28}$$

□

Lemma 3.9. *For $m = 3$, four analytic-Zariski open incidence charts in the unramified locus of $\bar{Y}_{m,i}$ have the exact form (3.8). Their union is invariant under the sign involution and contains both complete curves in $\bar{p}_i^{-1}(C_i)$.*

Proof. Write $\zeta^3 = 1$ and, on (3.26), set

$$d = \lambda^2 + \zeta^2 = x, \qquad K = 2\lambda^3 + 3\zeta^2\lambda. \tag{3.29}$$

Expansion using $\zeta^3 = 1$ gives

$$K^2 = 4d^3 - 3\zeta d - 1. \tag{3.30}$$

The first base chart is parametrized by all of $\mathbf{C}_\lambda^* \times \mathbf{C}_u$:

$$x = d, \quad y = K + du, \quad z = \zeta - \frac{u(2K + du)}{3}. \tag{3.31}$$

Substitution in E_3 proves that it maps into the double cover. For $d \neq 0$ its inverse is $u = (y - K)/d$. When $d = 0$, the condition $\lambda \neq 0$ gives $K \neq 0$. On the component $y = K$ the identity

$$(y - K)(y + K) = -3d(z - \zeta) \tag{3.32}$$

extends that inverse as $u = -3(z - \zeta)/(y + K)$. Thus (3.31) is an open embedding with image obtained by deleting $\lambda = 0$ and the other component $\{d = 0, y = -K\}$ from the affine double cover. Its sign-conjugate chart is

$$y = -K + du', \qquad z = \zeta - \frac{u'(-2K + du')}{3}, \qquad u' = u + \frac{2K}{d} \tag{3.33}$$

on their overlap $d \neq 0$.

The line bundle L_m is holomorphically trivial on either base chart: it is Stein and homotopy equivalent to a circle, so its $H^2(-, \mathbf{Z})$, and hence its Picard group by the exponential sequence, vanishes. With $Q_\zeta(z) = z^2 + \zeta z + \zeta^2$, (3.31) gives

$$\Delta_3(z) = uG(\lambda, u), \qquad G = -\frac{1}{3}(2K + du)Q_\zeta(z). \tag{3.34}$$

The full incidence chart is therefore

$$u = Ap, \qquad B = pG(\lambda, Ap), \quad (\lambda, A, p) \in \mathbf{C}^* \times \mathbf{C}^2. \tag{3.35}$$

If $A = 0$, then $u = 0$, hence $z = \zeta$. This chart encounters no node at another cusp and no exceptional curve removed in $Y_{m,i}$. Its only possible exceptional affine lines are

$$\{\lambda = \lambda_\pm,\ A = 0,\ p \in \mathbf{C}\}. \tag{3.36}$$

Indeed $u = 0$ is a node exactly when $K = 0$, and the solutions with $\lambda \neq 0$ are (3.28). There $K' = 6\lambda^2 + 3\zeta^2 = -6\zeta^2 \neq 0$. Together with $d = -\zeta^2/2 \neq 0$, this shows that u, u' are transverse coordinates at either lifted node. Near it the two factors are related by

$$\Delta_3 = h\,uu', \qquad h = -dQ_\zeta(z)/3, \qquad h(e_\zeta) = \zeta^4/2 \neq 0. \tag{3.37}$$

At one lifted node choose the sign so that (3.35) has the ruling of the actual small resolution. Its complementary chart, defined on the entire sign-conjugate base chart, has

$$u' = Br, \qquad A = rG'(\lambda, u'), \qquad G' = -\frac{1}{3}(-2K + du')Q_\zeta(z). \tag{3.38}$$

On the common exceptional neighbourhood its coordinates satisfy $r = (hp)^{-1}$, by (3.37). The two charts therefore cover the whole first lifted $\mathbf{P}^1$, including the point $p = \infty$. Applying the sign involution gives the two charts covering the other lifted $\mathbf{P}^1$. For each chart, Lemma 3.7 deletes at most the other lifted node's entire affine p-line if its ruling is opposite. Hence its image is precisely (3.8), with no restriction on p along a deleted line. The base-chart complements and the strict-transform divisors in Lemma 3.7 are closed analytic, so each image is analytic-Zariski open. □

Lemma 3.10. *For $m = 4$, four analytic-Zariski open incidence charts in the unramified locus of $\bar{Y}_{m,i}$ have the exact intrinsic form* (3.9)*. Their union is invariant under the sign involution and contains both complete curves in $\bar{p}_i^{-1}(C_i)$.*

Proof. Let $\zeta^4 = 1$ and use d, K from (3.29). Direct expansion gives

$$4d^3 - 3d = K^2 + \zeta^2. \tag{3.39}$$

On the double cover of the affine part of E_4, the open conditions $\lambda \neq 0$ and $t = y + iz \neq 0$ give exactly $T = (\mathbf{C}^*)^2_{\lambda,t}$, since

$$y = \frac{1}{2}\left(t + \frac{K^2 + \zeta^2}{t}\right), \qquad z = \frac{1}{2i}\left(t - \frac{K^2 + \zeta^2}{t}\right). \tag{3.40}$$

Its complement consists of the algebraic loci O, $\lambda = 0$, and $t = 0$. No trivialization of $L_m|_T$ is used. Set

$$q_- = i\zeta - K, \qquad q_+ = i\zeta + K. \tag{3.41}$$

Equation (3.40) yields the exact factorization

$$z - \zeta = \frac{(t - q_-)(t - q_+)}{2it}, \qquad \Delta_4 = (t - q_-)G_4 = (t - q_+)G_4^+, \tag{3.42}$$

where both G_4, G_4^+ are holomorphic on T. The first full incidence chart is

$$\langle A, p\rangle = t - q_-(\lambda), \qquad B = pG_4(\lambda, t), \qquad A \in L_m|_T, \quad p \in (L_m|_T)^\vee. \tag{3.43}$$

It is exactly $Z(L, q)$ in (3.9), with $q(\lambda) = i\zeta - 2\lambda^3 - 3\zeta^2\lambda$. In particular, it is the full twisted model over T. At $A = 0$, its incidence equation forces $t = q_-$; then (3.42) gives $z = \zeta$. Thus it avoids all other removed exceptional curves.

The two nodes over e_ζ occur at $\lambda = \lambda_\pm$, $t = i\zeta = q_-(\lambda_\pm)$. Here $K'(\lambda_\pm) = -6\zeta^2 \neq 0$, so $t - q_-$ and $t - q_+$ have independent differentials. Moreover

$$\Delta_4 = h(t - q_-)(t - q_+), \qquad h(e_\zeta) = 2\zeta^2 \neq 0. \tag{3.44}$$

The exceptional part of (3.43) over either node is the complete affine line

$$\{\lambda = \lambda_\pm,\ t = i\zeta,\ A = 0,\ p \in (L_m|_T)^\vee\}. \tag{3.45}$$

The complementary ruling chart is defined on the whole of T by $Br = t - q_+$, $A = rG_4^+$. Over the node where the first chart matches the actual ruling, the two exceptional coordinates satisfy $r = (hp)^{-1}$, so they cover the complete lifted $\mathbf{P}^1$. The sign involution $\lambda \mapsto -\lambda$ interchanges q_- and q_+ and gives the other two charts, covering the second lifted $\mathbf{P}^1$. By Lemma 3.7, the only possible deletions are entire lines of (3.45) at opposite-ruling nodes. They have $\lambda_\pm \neq 0$ and $q_\pm(\lambda_\pm) = i\zeta \neq 0$. Each chart image is analytic-Zariski open: its complement is the inverse image of the algebraic base-chart complement together with finitely many strict-transform divisors. This proves the exact form (3.9) without assuming $L_m|_T$ trivial. □

Theorem 3.11. *For every $a_* \in \mathcal{A}$, the constructed member (X_{a_*}, f_{a_*}) has the conic-band package* (G1)–(G4) *of Definition 3.4. The finite and cyclic coverings in* (G1) *may be chosen to be those of Equation (3.1) and Lemma 3.3.*

Proof. Lemma 3.5 realizes the two covers, their analytic-Zariski band cover, the line bundle, and the proper small resolution. Lemma 3.6 identifies its base with the required two-chart algebraic surface, completing (G1)–(G2). Lemma 3.8 constructs the global finite covers in (G3).

For a fixed exceptional curve C_i, let $\bar{N}_i$ be the union of the four charts constructed in Lemmas 3.9 and 3.10, using the lemma for the relevant value of m. Each chart is analytic-Zariski open in $\bar{Y}_{m,i}$, so the complement of their finite union is the intersection of four closed analytic sets and is closed analytic. All four charts lie in $\lambda \neq 0$, the unramified locus of the double cover. The sign transforms make their union invariant, while the two reciprocal chart pairs contain both complete components of $\bar{p}_i^{-1}(C_i)$. These observations give all the invariance, branch-avoidance, and complement assertions in (G3). The precise chart descriptions and complete-line deletions are (G4). The parameter a_* was arbitrary. It changes the scaling cocycle of L_m but not the elliptic base, the local node factors, or the extent of these charts. □

Thus the geometry used below is established on the actual coverings of every member X_a. The remaining task is to prove Oka flexibility on the realized charts and descend it through the coverings.

3.4. **Oka properties of the complements and relative descent.** A manifold is *elliptic* if it admits a holomorphic spray on a vector bundle whose differential along the zero section surjects onto the tangent bundle. By Gromov's theorem [1], elliptic manifolds are Oka. We use the following elementary sufficient criterion. All complete fields below are complete for every complex time.

Lemma 3.12. *If finitely many complete holomorphic vector fields $V_1, \dots, V_N$ span T_zZ at every $z \in Z$, then Z is elliptic and hence Oka.*

Proof. Let Φ_j^t be the entire flow of V_j. The map

$$s : Z \times \mathbb{C}^N \longrightarrow Z, \qquad s(z, t_1, \dots, t_N) = \Phi_N^{t_N} \circ \cdots \circ \Phi_1^{t_1}(z)$$

is a holomorphic spray. Its differential in the parameter directions at $(z, 0)$ has image

$$\operatorname{span}\{V_1(z), \dots, V_N(z)\} = T_zZ.$$

□

Lemma 3.13. *Let V be a complete holomorphic vector field on a complex manifold Z, with entire flow $\Phi : \mathbb{C} \times Z \to Z$. If $h \in \mathcal{O}(Z)$ is a first integral of V, that is $V(h) = 0$, then hV is complete and its flow is*

$$\Psi_t(z) = \Phi_{th(z)}(z), \qquad t \in \mathbb{C}.$$

If h vanishes on a subset $A \subset Z$, this flow fixes A pointwise.

Proof. Along an orbit of V, the function h is constant. Therefore

$$\frac{\partial}{\partial t}\Phi_{th(z)}(z) = h(z)V\big(\Phi_{th(z)}(z)\big) = h\big(\Phi_{th(z)}(z)\big)V\big(\Phi_{th(z)}(z)\big).$$

Thus Ψ_t is the flow of hV and is defined for every complex time. If $h(z) = 0$, then $\Psi_t(z) = \Phi_0(z) = z$. □

Lemma 3.14. *Let $f \in \mathbb{C}[c]$ have only simple roots. Then the smooth affine surface*

$$W_f = \{(a, b, c) \in \mathbb{C}^3 : ab = f(c)\}$$

has tangent bundle spanned by finitely many complete holomorphic vector fields and is therefore elliptic.

Proof. The gradient of $ab - f(c)$ can vanish on the hypersurface only if $a = b = 0$, $f(c) = 0$, and $f'(c) = 0$, which is excluded by the hypothesis. The following tangent vector fields are complete:

$$D_a = a\partial_c + f'(c)\partial_b, \qquad D_b = b\partial_c + f'(c)\partial_a, \qquad H = a\partial_a - b\partial_b.$$

For example, the flow of D_a fixes a, sends c to $c + sa$, and sends b to

$$b + \frac{f(c + sa) - f(c)}{a};$$

the quotient is a polynomial with its removable value at $a = 0$. The D_b flow is analogous and the H flow is $(a, b, c) \mapsto (e^s a, e^{-s} b, c)$. If $a \neq 0$, the fields D_a, H span the two-dimensional tangent space; if $b \neq 0$, use D_b, H. If $a = b = 0$, then c is a simple root of f and $D_a = f'(c)\partial_b$, $D_b = f'(c)\partial_a$, which again span. The composition of their flows is a dominating spray. □

Lemma 3.15. *Let Q carry finitely many complete holomorphic fields $W_1, \dots, W_N$ spanning TQ, and let $g \in \mathcal{O}(Q)$. Then the regular locus of*

$$H_g = \{\alpha\beta = g(q)\} \subset Q \times \mathbf{C}^2$$

is elliptic.

Proof. For every j set

$$X_j = \alpha W_j + (W_j g)\partial_\beta, \qquad Y_j = \beta W_j + (W_j g)\partial_\alpha, \qquad E = \alpha\partial_\alpha - \beta\partial_\beta.$$

They are tangent and complete. For example the X_j flow fixes α, moves q by the W_j flow for time $s\alpha$, and changes β by the corresponding difference quotient of g. Their flows are automorphisms of H_g and therefore preserve its singular set and its regular locus. All fields below are consequently complete on H_g^{reg}, not merely on the ambient hypersurface.

If $\alpha \neq 0$, the X_j together with E span; if $\beta \neq 0$, use the Y_j. It remains to consider $\alpha = \beta = 0$ and $dg \neq 0$. Include all complete conjugates

$$Z_{jk} = (\mathrm{Fl}_1^{X_j})_* Y_k.$$

Write the point in question as $a = (q, 0, 0)$. The time-minus-one flow sends it to

$$a_- := \mathrm{Fl}_{-1}^{X_j}(a) = (q, 0, -(W_j g)(q)).$$

Since the base component of the time-one flow is $\mathrm{Fl}_\alpha^{W_j}(q)$, its differential at a_- has base component

$$(\delta q, \delta\alpha, \delta\beta) \longmapsto \delta q + \delta\alpha\, W_j(q).$$

At a_-, the base and α components of Y_k are $-(W_j g)(q) W_k(q)$ and $(W_k g)(q)$, respectively. Pushing forward therefore gives the base component

$$(W_k g) W_j - (W_j g) W_k$$

of Z_{jk} at a. To see the spanning assertion explicitly, choose j_0 with $c := W_{j_0} g \neq 0$. If $v = \sum_k a_k W_k \in \ker dg$, then

$$\sum_k a_k \big((W_k g) W_{j_0} - (W_{j_0} g) W_k \big) = -cv.$$

Thus these vectors span $\ker dg$. The corresponding X_{j_0} and Y_{j_0} give the two vertical directions. A finite family therefore dominates the regular locus. □

Lemma 3.16. *For the package of Definition 3.4, the manifold U_m^{reg} is Oka.*

Proof. Every point of E_m has an algebraic Zariski open neighbourhood $Q \simeq \mathbf{C}^* \times \mathbf{C}$. Indeed, in either Danielewski chart the complete fields of Lemma 3.14 have algebraic time maps. Their spanning property makes all orbits of the generated group open. The surface is connected, so this group is transitive. An automorphism carrying the chosen point into $\{a \neq 0\}$ pulls that set back to the required Q.

The Stein manifold Q retracts onto a circle. By the exponential sequence and Cartan's Theorem B, $\mathrm{Pic}(Q) \simeq H^2(Q, \mathbf{Z}) = 0$, so $L_m|_Q$ has a holomorphic frame. In this frame the conic space over Q is the suspension $\{AB = g(q)\}$ with $g = \Delta_m \circ z$. The complete fields $a\partial_a, \partial_c$ span the tangent bundle of $Q \simeq \mathbf{C}_a^* \times \mathbf{C}_c$. By Lemma 3.15, the regular locus of this suspension is elliptic. It is analytic-Zariski open in H_m^{reg}: its complement is the intersection of that regular locus with the inverse image of the algebraic complement of Q in E_m. These open sets cover H_m^{reg}. Localization in Theorem 3.2 makes H_m^{reg} Oka, and the resolution identifies it with U_m^{reg}. □

We now prove the flexibility of the exact models in Definition 3.4. Fix $q \in \mathcal{O}(\mathbf{C}^*)$ and $\lambda_1, \dots, \lambda_s \in \mathbf{C}^*$ with $q(\lambda_j) \neq 0$.

Let $T = (\mathbf{C}^*)^2$ have coordinates (λ, t), let $L \to T$ be any holomorphic line bundle, and let $L^\vee$ be its dual. Set

$$Z(L, q) = \{(\lambda, t, A, p) : A \in L_{(\lambda,t)},\ p \in L^\vee_{(\lambda,t)},\ \langle A, p\rangle = t - q(\lambda)\}. \tag{3.46}$$

For each j let

$$\ell_j(L) = \{\lambda = \lambda_j,\ t = q(\lambda_j),\ A = 0\} \subset Z(L, q);$$

the dual coordinate p is unrestricted there.

Lemma 3.17. *The manifold $Z(L, q) \setminus \bigcup_j \ell_j(L)$ is Oka.*

Proof. Pull (3.46) back by the universal covering $(x, y) \mapsto (e^x, e^y)$ of T. The pulled-back line bundle is trivial: the exponential sequence and Cartan's Theorem B identify its Picard group with $H^2(\mathbf{C}^2, \mathbf{Z}) = 0$. Writing $\lambda = e^x$ and $t = e^y = Ap + q(\lambda)$ gives the unramified cover

$$\mathcal{H} = \{Ap = f(x, y) := e^y - q(e^x)\} \subset \mathbf{C}^4. \tag{3.47}$$

The deleted lifts are

$$L_j = \{e^x = \lambda_j,\ e^y = q(\lambda_j),\ A = 0,\ p \in \mathbf{C}\}.$$

Set

$$c(x) = \prod_j (e^x - \lambda_j), \qquad f_x = -e^x q'(e^x), \qquad f_y = e^y.$$

Consider the complete tangent fields

$$\begin{aligned} X &= A\partial_x + f_x \partial_p, & X_y &= A\partial_y + f_y \partial_p, \\ Y &= p\partial_y + f_y \partial_A, & V &= cY, \\ E &= A\partial_A - p\partial_p. \end{aligned}$$

The usual difference-quotient formulas give completeness of X, X_y, Y. For example, the X-flow fixes A, y, replaces x by $x + sA$, and replaces p by

$$p + \frac{f(x + sA, y) - f(x, y)}{A},$$

where the quotient has its holomorphic limiting value when $A = 0$. The formulas for X_y and Y are identical with the corresponding variables interchanged. Since $Y(c) = 0$, Lemma 3.13 makes $V = cY$ complete. The fields X, X_y, E preserve every L_j, while V fixes them pointwise. Let ϕ be the time-one flow of X and set $Z = \phi_* V$. Then Z is complete and preserves the deleted set. The hypersurface $\mathcal{H}$ is smooth because $f_y = e^y$ never vanishes.

If $A \neq 0$, the fields X, X_y, E span. If $A = 0$ outside the deleted lines, then $f = 0$ and necessarily $cf_y \neq 0$. There

$$X_y = f_y \partial_p, \qquad V = c(p\partial_y + f_y \partial_A).$$

The inverse X flow replaces p by $p - f_x$, and, modulo ∂_p,

$$Z = c\big(f_y \partial_x + (p - f_x)\partial_y + f_y \partial_A\big).$$

Thus

$$Z - V \equiv c(f_y \partial_x - f_x \partial_y) \quad (\mathrm{mod}\ \partial_p),$$

a nonzero tangent direction to $f = 0$. Hence X_y, V, Z span. The complement of the lifted lines is elliptic. Covering invariance in Section 3 descends the Oka property through the unramified exponential covering. □

Lemma 3.18. *Let $w_1, \ldots, w_s \in \mathbf{C}^*$. The corresponding order-three product complement*

$$(\mathbf{C}^*_w \times \mathbf{C}_A \times \mathbf{C}_p) \setminus \bigcup_j \{w = w_j, A = 0\}$$

is Oka.

Proof. On the exponential cover $(x, A, p) \mapsto (e^x, A, p)$, set $c(x) = \prod_j (e^x - w_j)$ and consider the complete fields

$$P = c\partial_A, \qquad Q = A\partial_x, \qquad H = A\partial_A, \qquad R = \partial_p,$$

together with $Z = (\mathrm{Fl}_1^P)_* Q$. All preserve the deleted set: P vanishes there, while Q, H, R are tangent to it; hence so is Z. If $A \neq 0$, the fields Q, H, R span. If $A = 0$ outside the deleted set, then $c(x) \neq 0$; the field P gives the A direction, R gives the p direction, and Z has nonzero x component $-c(x)\partial_x$ (additional A components are irrelevant). Thus the lifted complement is elliptic, and covering invariance in Section 3 descends Oka through the exponential covering. □

The incidence charts descend through the finite quotients.

Lemma 3.19. *Let a finite group G act holomorphically on a normal complex space $\bar{Z}$, and let $p : \bar{Z} \to Z \simeq \bar{Z}/G$ be the finite quotient to a complex manifold. Let $D, C \subset Z$ be disjoint closed analytic subsets, with p unramified over $Z \setminus D$. Suppose that a G-invariant open submanifold $\bar{N} \subset p^{-1}(Z \setminus D)$ contains $p^{-1}(C)$ and has closed analytic complement in $\bar{Z}$. There is an analytic-Zariski open set $N \subset Z \setminus D$ containing C such that $p^{-1}(N) = \bar{N}$. The map $p : \bar{N} \to N$ is a finite unramified covering. If $\bar{N}$ is Oka, then N is Oka.*

Proof. Set $A = \bar{Z} \setminus \bar{N}$ and $N = Z \setminus p(A)$. The set $p(A)$ is closed analytic by Remmert's proper mapping theorem. Each fibre of p is one G-orbit. Invariance therefore forces each fibre to lie entirely in $\bar{N}$ or entirely in A, giving $p^{-1}(N) = \bar{N}$. The assumed inclusions imply $C \subset N$ and $N \cap D = \emptyset$. Thus the restriction of p is finite and unramified between manifolds, so is a holomorphic covering. The last assertion is covering invariance. □

Proposition 3.20. *For the package of Definition 3.4, both U_1 and U_2 are Oka. Consequently X_{a_*} is Oka.*

Proof. Fix m. By Lemma 3.16, the open manifold U_m^{reg} is Oka. For each exceptional curve C_i, the four incidence charts covering $\bar{N}_i$ are Oka by Lemmas 3.17 and 3.18. They are analytic-Zariski open in $\bar{N}_i$, so localization makes $\bar{N}_i$ Oka. Apply Lemma 3.19 to $\bar{p}_i$ to obtain an analytic-Zariski open Oka neighbourhood N_i of C_i in $Y_{m,i}$. The two analytic-Zariski open sets N_i and U_m^{reg} cover $Y_{m,i}$; hence $Y_{m,i}$ is Oka. The sets $Y_{m,i}$ cover Y_m and are analytic-Zariski open in it. If there are no exceptional curves, $Y_m = U_m^{\mathrm{reg}}$ instead. In both cases Y_m is Oka.

The band cover in (G1) now makes $\widehat{N}_m$ Oka by localization. The two unramified coverings descend this property to N_m and then to the corresponding U_j. Doing this for $m = 4$ and $m = 3$ proves the two assertions. Finally, U_1, U_2 are analytic-Zariski open and cover X_{a_*}, so another localization gives its Oka property. □

The Oka properties of the two open complements now give the relative Oka-map theorem.

Proof of Theorem 1.1. For $(m, j) = (4, 1)$ and $(3, 2)$, at a_* the composite

$$Q_{m,a_*} \xrightarrow{q_{m,a_*}} M_{m,a_*} \xrightarrow{r_{m,a_*}} U_{j,a_*}$$

is a surjective unramified holomorphic covering. Oka manifolds are invariant under such coverings. The geometric realization and Oka criterion in Theorem 3.11 and Proposition 3.20 therefore make Q_{m,a_*} Oka for every $a_* \in \Delta$.

By Lemma 3.3, the composite covering can be written

$$P_m : Q_{m,a_*} \times \Delta \longrightarrow \mathcal{U}_j, \qquad \pi \circ P_m = \mathrm{pr}_\Delta . \tag{3.48}$$

This gives the needed relative Oka property; the following spray argument makes the descent explicit.

Let $K \subset \mathbf{C}^n$ be compact and convex, and let $f : \mathrm{Op}\, K \to \mathcal{U}_j$ be holomorphic. Shrink its domain to a convex neighbourhood of K. Because this domain is simply connected, f lifts holomorphically through (3.48) to

$$\widetilde{f} = (g, \alpha) : \mathrm{Op}\, K \longrightarrow Q_{m,a_*} \times \Delta, \qquad \alpha = \pi \circ f.$$

Theorem 1.3 of [24], applied to the map $Q_{m,a_*} \to \{*\}$, gives, after a further shrinking if necessary, a global-parameter spray

$$\sigma : \mathrm{Op}\, K \times \mathbf{C}^N \longrightarrow Q_{m,a_*}, \qquad \sigma(z, 0) = g(z),$$

whose parameter differential at zero is onto $T_{g(z)}Q_{m,a_*}$ for $z \in K$. Set

$$s(z, w) = P_m\big(\sigma(z, w), \alpha(z)\big).$$

Then $s(z, 0) = f(z)$ and $\pi \circ s(z, w) = \alpha(z)$. The covering P_m is locally biholomorphic, so its differential identifies the vertical tangent spaces. Thus s is a dominating global π-spray over f.

It follows that $\pi|_{\mathcal{U}_j}$ is convexly elliptic, hence satisfies POPAI by [24, Theorem 1.3]. As $\mathcal{U}_1, \mathcal{U}_2$ are an analytic-Zariski open cover of the total space, the localization theorem [24, Corollary 5.13] gives POPAI for π.

Finally, π is a surjective proper holomorphic submersion by the relative compactification theorem in Section 2. Ehresmann's theorem makes it a smooth locally trivial fibre bundle, hence a Serre fibration. A holomorphic submersion which is a Serre fibration and satisfies POPAI is an Oka map. Since a_* was arbitrary, Theorem 3.11 and Proposition 3.20 also give the asserted Oka property of both open complements and of X_a for every $a \in \Delta$. □

4. Finite-point and curve complements

4.1. **Analytic tools and geometry.** For the point-modification proofs, fix $a \in \mathcal{A}$, abbreviate $X = X_a$, $S_j = S_{j,a}$, $U_j = X \setminus S_j$, and write $M_m = N_m$, $\widehat{M}_m = Q_{m,a}$ for the finite and cyclic covers of Lemma 3.3 and Theorem 3.11. The map $\pi_m : \widehat{M}_m \to M_m$ has residual deck group $\mathbf{Z}$. Every centre below is its *complete* inverse image under each covering. The cyclic cover need not be simply connected, and none of the following arguments uses simple connectedness.

We collect the external results used in the proof. They are stated here so that later applications need only verify their hypotheses.

An open subset V of a complex manifold Z is *analytic-Zariski open* if $Z \setminus V$ is a closed complex-analytic subvariety of Z. Analyticity here is always relative to the displayed ambient manifold. In particular, after deleting a closed discrete set S, an analytic subset of $Z \setminus S$ has the form $A \cap (Z \setminus S)$ locally, for an analytic subset A of Z.

Theorem 4.1. *Let $\pi : \widetilde{Z} \to Z$ be a surjective unramified holomorphic covering of complex manifolds, with an arbitrary number of sheets. Then $\widetilde{Z}$ is Oka if and only if Z is Oka* [20, Proposition 5.6.3].

Theorem 4.2. *If S is a Stein manifold and $\mathcal{F}$ is a coherent analytic sheaf on S, then*

$$H^q(S, \mathcal{F}) = 0 \qquad (q \geq 1)$$

[20, Theorem 2.6.1].

Lemma 4.3. *Let S be a Stein manifold which has the homotopy type of a circle. Then every holomorphic line bundle on S is holomorphically trivial.*

Proof. The exponential sequence and Cartan's Theorem B give $\mathrm{Pic}(S) \simeq H^2(S, \mathbb{Z})$, which vanishes because S has the homotopy type of a circle. □

Lemma 4.4. *A finite, surjective, unramified holomorphic map between complex manifolds is a finite-sheeted holomorphic covering.*

Proof. Let $f : Z \to W$ be such a map and fix $w \in W$. The fibre $f^{-1}(w)$ is finite. By the holomorphic inverse function theorem, its points have pairwise disjoint neighbourhoods which map biholomorphically onto neighbourhoods of w. Shrinking their target neighbourhoods to a common open set $V \ni w$ gives the expected inverse branches. There can be no additional branch above arbitrarily small neighbourhoods of w. Otherwise, choose points on additional branches mapping to a sequence tending to w. Properness gives a convergent subsequence upstairs; its limit would be an additional point of $f^{-1}(w)$, contrary to the list of inverse branches. Hence V is evenly covered. □

We next state the two flexibility results for complements in affine space which drive all logarithmic-cover arguments.

Definition 4.5. An infinite closed discrete set $E \subset \mathbb{C}^n$ is *Rosay–Rudin tame* if a holomorphic automorphism of $\mathbb{C}^n$ carries it onto the standard sequence $\{(j, 0, \dots, 0) : j \in \mathbb{N}\}$. Finite subsets are called tame as well. A positive-dimensional closed analytic subvariety $A \subset \mathbb{C}^n$ is *tame* in the sense used below if an automorphism of $\mathbb{C}^n$ carries it to a subvariety whose projective closure in $\mathbb{P}^n$ does not contain the hyperplane at infinity.

Definition 4.6. A holomorphic spray on a complex manifold Z is a holomorphic map $s : E \to Z$ from the total space of a holomorphic vector bundle $E \to Z$ such that $s(0_z) = z$ for every $z \in Z$. It is *dominating* if the differential of s at 0_z, restricted to the vertical tangent space E_z, maps surjectively onto $T_z Z$ for every $z \in Z$. The manifold Z is *elliptic* if it admits a dominating holomorphic spray.

Theorem 4.7. *Let $E \subset \mathbb{C}^{k+\ell} = \mathbb{C}^k \times \mathbb{C}^\ell$ be an infinite closed discrete set, where $k, \ell \geq 1$. If the image of E under the coordinate projection to $\mathbb{C}^k$ is closed and discrete, then E is tame. The projection is not required to have finite fibres* [26, Theorem 3.9].

Theorem 4.8. *Let $A \subset \mathbb{C}^n$ be a tame closed analytic subvariety with* $\dim A \leq n - 2$. *Then $\mathbb{C}^n \setminus A$ is elliptic* [23, Proposition 2.24], *and hence Oka by Gromov's theorem* [1]. *In particular, the conclusion applies to a tame discrete subset of $\mathbb{C}^3$ and to a tame curve in $\mathbb{C}^3$.*

For a positive-dimensional A, an automorphism first carries A to the projectively tame position in Forstnerič's proposition; the result then pulls back to the original complement. For an infinite Rosay–Rudin tame discrete set, the defining automorphism sends it to the standard sequence, to which the same proposition applies.

For later interpolation we use the following standard consequence of Cartan's Theorem B.

Theorem 4.9. *Let Z be a Stein manifold and let $E \subset Z$ be closed and discrete. Every function $a : E \to \mathbb{C}$ is the restriction of a holomorphic function $h \in \mathcal{O}(Z)$.*

Proof. Cartan's Theorem B applied to the coherent ideal sheaf of E makes $\mathcal{O}(Z) \to \mathcal{O}(E)$ surjective. Since E is reduced and discrete, $\mathcal{O}(E)$ consists of all functions $E \to \mathbb{C}$. □

We shall also prescribe zeros of an entire function.

Theorem 4.10. *Let $A \subset \mathbb{C}$ be a closed discrete set and attach to each $a \in A$ a positive integer m_a. There is an entire function $h \in \mathcal{O}(\mathbb{C})$ whose zero at each a has multiplicity m_a and which has no other zeros. In particular, there is a nonzero entire function vanishing on A* [25, Theorem 26.5].

Theorem 4.11. *The complex threefold X has the following structure.*

(G1) *The reduced multiple fibres S_1, S_2 give $U_j = X \setminus S_j$ and $U_1 \cup U_2 = X$. The finite unramified covers of* Equation (3.1) *and* Theorem 3.11 *are $\varpi_4 : M_4 \to U_1$ and $\varpi_3 : M_3 \to U_2$, of degrees four and three.*

(G2) *For $m = 3, 4$, the cyclic covering*

$$\pi_m : \widehat{M}_m \longrightarrow M_m$$

has residual deck group $\Gamma_m = \langle \gamma_m \rangle \simeq \mathbb{Z}$. The manifold $\widehat{M}_m$ has an analytic-Zariski-open cover by the translates

$$Y_m^{(r)} = \gamma_m^r(Y_m), \qquad r \in \mathbb{Z},$$

of a two-row band Y_m. Every band contains the same principal smooth core

$$\mathcal{K}_m = \{\Delta_m(z) \neq 0\}, \qquad \Delta_3(z) = z^3 - 1, \qquad \Delta_4(z) = 1 - z^4.$$

The complement of this core consists of toroidal central-fibre strata. On the common core, the deck generator preserves the coordinate z. The toroidal strata carry an integer row index b such that the chosen generator γ_m sends b to $b-1$, while the toroidal part of the fixed two-row band Y_m consists of the rows $b = 0$ and $b = 1$. Consequently, the nth translate of a toroidal point has row index $b - n$, and membership in Y_m forces $b - n \in \{0, 1\}$.

(G3) *The surface E_m, its fibration $\beta_m : E_m \to \mathbf{C}_z$, the two disjoint sections O, P_m, and the line bundle $\mathcal{L}_m$ are those of Definition 3.4 and Theorem 3.11. The two Danielewski charts, exchanged by a fibrewise algebraic automorphism, have the equations and regular algebraic base functions in (3.6)–(3.7). Their defining polynomials have simple roots. Every reduced level $\{z = z_0\}_{\mathrm{red}} \subset E_m$ is therefore a closed algebraic curve; its intersection with an algebraic-Zariski-open subchart is a finite union of closed reduced algebraic curves there.*

On the total space of $\mathcal{L}_m \oplus \mathcal{L}_m^{-1}$, let

$$\mathcal{X}_m = \{A \otimes B = \Delta_m(z)\}, \qquad \Delta_3(z) = z^3 - 1, \quad \Delta_4(z) = 1 - z^4. \tag{4.1}$$

The proper small resolution $\rho_m : Y_m \to \mathcal{X}_m$ is the one already realized in Theorem 3.11; it is a biholomorphism from $U_m^{\mathrm{reg}} = Y_m \setminus \bigcup_i C_i$ onto $\mathcal{X}_m^{\mathrm{reg}}$. The vector-bundle projection gives the base projection $U_m^{\mathrm{reg}} \to E_m$, and the common principal core is its inverse image of $\{\Delta_m(z) \neq 0\}$.

(G4) *For each exceptional curve C_i, let $Y_{m,i} = U_m^{\mathrm{reg}} \cup C_i$ and use the normal Galois double cover $\bar{p}_i : \bar{Y}_{m,i} \to Y_{m,i}$ constructed in Lemma 3.8, with sign group $G_i = \{1, \tau_i\}$. Its bad divisor $D_{m,i}$ is disjoint from C_i, and the cover is unramified outside that divisor. In the source base notation, let*

$$\mathrm{pr} : Y_{m,i} \longrightarrow E_m$$

be the base projection induced by ρ_m and the vector-bundle projection $\mathcal{X}_m \to E_m$. Then

$$D_{3,i} = \mathrm{pr}^{-1}\big(O \cup \{a = r_i\}\big), \qquad D_{4,i} = \mathrm{pr}^{-1}\big(O \cup \{c = c_{1,i}\} \cup \{a = 0\}\big).$$

If $m = 3$ and C_i lies over the base root $z = \zeta_i$, where $\zeta_i^3 = 1$, then $r_i = \zeta_i^2$. If $m = 4$ and C_i lies over $z = \zeta_i$, where $\zeta_i^4 = 1$, then $c_{1,i} = \zeta_i^2$. Here O is the zero section; in order three $a = r_i$ is the exact quadratic branch divisor, while in order four $c = c_{1,i}$ is the exact branch divisor and $a = 0$ is the omitted-value divisor of the affine incidence coordinates.

The G_i-invariant set $\bar{N}_i$ of Theorem 3.11 contains $\bar{p}_i^{-1}(C_i)$, lies over $Y_{m,i} \setminus D_{m,i}$, and has closed analytic complement. Its four corrected incidence charts are analytic-Zariski open in $\bar{N}_i$; the two selected charts cover the standard affine charts at one lifted curve, and their sign transforms cover those at the other lift.

(G5) *Fix a cube root of unity ζ, so $\zeta^3 = 1$. Every selected order-three affine chart is an open subset of the incidence model*

$$\mathbb{C}_w^* \times \mathbb{C}_A \times \mathbb{C}_p, \qquad R = Ap,$$

obtained by deleting the affine exceptional line corresponding to each wrong ruling. For a single selected lifted node there is exactly one such line,

$$\ell_j = \{w = w_j,\ A = 0,\ p \in \mathbb{C}\}, \qquad w_j \neq 0;$$

allowing a finite family of lifted nodes gives a finite union of these lines. The functions used below are

$$h(w) = 4w^3 + 6w, \qquad q(w) = 1 + w^2.$$

In the source affine parameter s, the selected factorization has

$$R = Ap = 2w - s, \qquad z - \zeta = \frac{\zeta}{3} R\big(2w^3 + 4w + (1 + w^2)s\big). \tag{4.2}$$

For a point over a common-principal-core fibre value z_ν, its full residual-deck orbit, whenever represented in this uncorrected product chart, is contained in the affine algebraic carrier

$$\mathscr{C}_\nu = \{(w, A, p) : R(h(w) - q(w)R) = c_\nu\}, \qquad c_\nu = \frac{3(z_\nu - \zeta)}{\zeta} \neq 0. \tag{4.3}$$

This identity holds in the uncorrected product model, before the wrong line is removed. The companion chart has the same form with its incidence coordinate denoted by B. The projective incidence coordinates are $p_A = \eta_+/\xi_+$ and $p_B = \xi_-/\eta_-$. The source identities are

$$p_B \circ \tau = -p_A, \qquad v_+ = Ap_A, \qquad u_- = Bp_B, \qquad \tau^* u_- = -v_+. \tag{4.4}$$

On the corresponding uncorrected product charts the sign transformation extends holomorphically and, after matching the companion notation, is

$$(w, A, p_A) \longmapsto (-w, B = A, p_B = -p_A). \tag{4.5}$$

(G6) *Fix a fourth root of unity ζ and set*

$$a_0 = i\zeta, \qquad H(h) = 2h^3 + 3\zeta^2 h, \qquad q(h) = a_0 - H(h).$$

Let $T = (\mathbb{C}^*)^2_{h,a}$. The incidence coordinate is intrinsically a section of a holomorphic line bundle $\mathcal{L} \to T$. A selected uncorrected chart is the twisted suspension

$$Z_4 = \{(h, a, A, p) : A \in \mathcal{L}_{(h,a)},\ p \in \mathcal{L}^\vee_{(h,a)},\ \langle A, p\rangle = a - q(h)\}. \tag{4.6}$$

Its corrected form deletes exactly one wrong-ruling affine line

$$\ell = \{(h_w, q(h_w), A = 0, p) : p \in \mathcal{L}^\vee_{(h_w, q(h_w))}\}, \qquad q(h_w) \neq 0.$$

For a common-principal-core fibre value z_ν, the base of the orbit lies on the algebraic carrier

$$(a - a_0 + H(h))(a - a_0 - H(h)) = 2ia(z_\nu - \zeta), \qquad z_\nu^4 \neq 1, \tag{4.7}$$

and the selected incidence product is

$$\langle A, p\rangle = a - a_0 + H(h).$$

In particular this product is nonzero on every such orbit and is bounded away from zero on compact parts of the finite carrier union. The companion chart exchanges the two factors in (4.7). The sign transformation exchanges the two lifted factorizations and the two rulings; it is an intrinsic bundle isomorphism and does not require a global frame of $\mathcal{L}$. On the exact common domain used to compare the two sign charts,

$$\kappa_\zeta = \frac{\Delta_4(z)}{z - \zeta}$$

is nonzero, a is nonzero, and $2ia/\kappa_\zeta$ is a holomorphic unit. In compatible projective coordinates the comparison is

$$[\xi_- : \eta_-] = \left[\eta_+ : \frac{2ia}{\kappa_\zeta}\xi_+\right].$$

Proof. The finite covers and the cyclic covers are constructed in Equation (3.1) and Lemma 3.3; their translated two-row bands and row action are obtained in Lemma 3.5. The algebraic surface, line bundle and resolved conic are identified in Lemma 3.6 and Theorem 3.11. The normalized double covers, their exact branch divisors and the four corrected charts are constructed in Lemmas 3.8 to 3.10. These statements apply for every $a \in \mathscr{A}$; the parameter changes the line-bundle cocycle but neither the two-row action nor the incidence identities. We record the carrier formulas used below. In order three, $s = 2w - R$ in (4.2) gives

$$2w^3 + 4w + (1 + w^2)s = 4w^3 + 6w - (1 + w^2)R = h(w) - q(w)R,$$

which proves (4.3). The projective-coordinate comparison in the two sign charts gives $(B \circ \tau)(p_B \circ \tau) = -Ap_A$ and $p_B \circ \tau = -p_A$. Thus $B \circ \tau = A$ where $p_A \neq 0$, and then everywhere by holomorphic continuation. This proves (4.5). In order four, the two factors of (4.7) are the two incidence factors of Equations (3.40) and (3.42); their comparison across the sign charts is Equation (3.43). In particular the incidence product does not vanish on a common-core orbit. The incidence coordinate is kept as a section of $\mathcal{L}$ throughout: no trivialization of this bundle over the full torus is used. ◻

The row calculation in (G2) has one immediate consequence which will be used repeatedly.

Lemma 4.12. *Every residual-deck orbit contained in the toroidal central-fibre part of $\widehat{M}_m$ meets a fixed two-row band in at most two points.*

Proof. Let b be the row index of the point. By Theorem 4.11(G2), the row index of its nth translate is $b - n$, and a two-row band contains only indices 0 and 1. Hence $b - n \in \{0, 1\}$, which has at most two integral solutions n. □

4.2. **Proper characters and complete lifts.** The local models used below are suspensions over a two-dimensional algebraic torus. Their deletion sets need not have any prescribed growth in a fixed Euclidean coordinate system. The substitute for such a growth condition is a monomial character that is proper on the finitely many algebraic curves carrying the infinite part of the deletion. We first establish this character and record the covering and projection facts that will be reused in the three local models.

Let

$$T = (\mathbf{C}^*)^2$$

with coordinates (u, v), and let $C_1, \dots, C_N \subset T$ be closed reduced algebraic curves. We always replace this family by the finite family of its irreducible components. For such a component C, let

$$\nu_C : \widetilde{C} \longrightarrow C$$

be its affine normalization, and let $\overline{\widetilde{C}}$ be the smooth projective completion of $\widetilde{C}$. At a boundary point $b \in \overline{\widetilde{C}} \setminus \widetilde{C}$, set

$$\mathbf{v}_b = \bigl(\mathrm{ord}_b(\nu_C^* u), \mathrm{ord}_b(\nu_C^* v)\bigr) \in \mathbf{Z}^2. \tag{4.8}$$

Lemma 4.13. *For every boundary point b as above, one has $\mathbf{v}_b \neq (0, 0)$.*

Proof. Suppose that both orders in (4.8) vanish. The two invertible functions u and v then extend holomorphically across b with nonzero values. Hence the map from a punctured disc about b to T extends to a map of the whole disc to T. Its image lies in C away from b, and therefore also at b, since C is closed in T. The valuative criterion for the finite normalization map now lifts the extended branch to $\widetilde{C}$. This contradicts $b \notin \widetilde{C}$. □

Lemma 4.14. *There is a nonzero vector $(r, s) \in \mathbf{Z}^2$ such that the character*

$$M(u, v) = u^r v^s \tag{4.9}$$

is proper and finite on the normalization of every irreducible component of $C_1 \cup \cdots \cup C_N$.

Proof. The projective completion of each normalized affine curve has only finitely many boundary points. Thus the family of boundary vectors (4.8), taken over all carrier components, is finite. Choose $(r, s) \in \mathbf{Z}^2$ outside the finitely many rational lines

$$r \,\mathrm{ord}_b\, u + s \,\mathrm{ord}_b\, v = 0. \tag{4.10}$$

This is possible because a finite union of rational lines does not exhaust $\mathbf{Z}^2$.

At every boundary point, the pullback of M has nonzero order. It therefore tends either to 0 or to ∞ along every branch approaching the boundary. If the inverse image of a compact subset of $\mathbf{C}^*$ were noncompact in a normalized carrier, its closure in the projective completion would meet the boundary, contradicting this behavior. Hence the restriction of M is proper. It is nonconstant, again because its order is nonzero at every boundary point. A fibre is both compact and discrete on the normalized curve, and is therefore finite. □

The character can be chosen subject to the additional linear conditions needed at a smooth suspension axis. Let

$$h(u, v) = \sum_{(i,j)\in\Lambda} c_{ij} u^i v^j \tag{4.11}$$

be a Laurent polynomial with distinct exponents and nonzero coefficients. For a vector (r, s), consider

$$W = s\,u\frac{\partial}{\partial u} - r\,v\frac{\partial}{\partial v}, \qquad \omega_{ij} = is - jr. \tag{4.12}$$

Lemma 4.15. *Let $q_e \in T$ satisfy $h(q_e) = 0$ and $dh_{q_e} \neq 0$. The vector (r, s) in Lemma 4.14 may be chosen so that*

$$(Wh)(q_e) \neq 0$$

and the integers ω_{ij}, $(i, j) \in \Lambda$, are pairwise distinct.

Proof. The equality $(Wh)(q_e) = 0$ is the linear condition

$$s(uh_u)(q_e) - r(vh_v)(q_e) = 0$$

on (r, s). It is a proper condition because $dh_{q_e} \neq 0$ and $u(q_e), v(q_e) \neq 0$. Equality of the weights belonging to two distinct exponents is the proper rational-line condition

$$(i - i')s - (j - j')r = 0.$$

Together with (4.10), these conditions exclude only finitely many rational lines. An integral vector outside their union has all the asserted properties. □

The wrong-ruling lines in the order-three charts occur over finitely many points on a common nonzero coordinate level. The following elementary variant records the separation used there.

Lemma 4.16. *Write the torus coordinates as (w, u), and let*

$$(w_1, u_0), \dots, (w_\ell, u_0) \in T, \qquad u_0 \neq 0,$$

where the w_j are pairwise distinct. One can choose $b \in \mathbf{Z}$ such that

$$M_b(w, u) = w\, u^b$$

is proper and finite on every carrier normalization. For every such b, the values $M_b(w_j, u_0)$ are pairwise distinct.

Proof. At a boundary point with valuation vector (α, β), properness fails only if

$$\alpha + b\beta = 0.$$

If $\beta = 0$, then $\alpha \neq 0$ by Lemma 4.13, so this equality never occurs. If $\beta \neq 0$, it excludes at most one integer b. There are only finitely many boundary points, and hence an integer avoiding all exclusions exists. The proof of Lemma 4.14 then gives properness and finiteness. At the common level $u = u_0$,

$$\frac{M_b(w_j, u_0)}{M_b(w_k, u_0)} = \frac{w_j}{w_k} \neq 1 \qquad (j \neq k),$$

so separation is automatic once the exponent of w is fixed to be one. □

Fix a nonzero character vector (r, s), and set

$$d = \gcd(|r|, |s|) > 0, \qquad K = \{(k, l) \in \mathbf{Z}^2 : rk + sl = 0\} = \mathbf{Z}\left(\frac{s}{d}, -\frac{r}{d}\right).$$

On the universal exponential cover

$$\exp : \mathbf{C}^2_{x,y} \longrightarrow T, \qquad (x, y) \longmapsto (e^x, e^y),$$

set

$$m = rx + sy. \tag{4.13}$$

Lemma 4.17. *The function m descends to the intermediate unramified cover*

$$T_M = \mathbf{C}^2/(2\pi i K) \longrightarrow T.$$

Its remaining deck group is infinite cyclic. If $k_0, l_0 \in \mathbf{Z}$ satisfy $rk_0 + sl_0 = d$, a generator acts by

$$[x, y] \longmapsto [x + 2\pi i k_0, y + 2\pi i l_0], \qquad m \longmapsto m + 2\pi i d.$$

Consequently, for any $q \in T$ and any logarithmic lift with value m_0, the set of m-values of the complete inverse image of q is exactly

$$m_0 + 2\pi i d\mathbf{Z}. \tag{4.14}$$

If an additional coordinate $A \in \mathbf{C}^*$ is written as $A = e^\sigma$, then the projection of the complete logarithmic inverse image of (q, A) to the (m, σ)-plane is

$$(m_0 + 2\pi i d\mathbf{Z}) \times (\sigma_0 + 2\pi i\mathbf{Z}). \tag{4.15}$$

No branch in the kernel lattice K is discarded: all such branches have the same m-value.

Proof. A deck translation $(x, y) \mapsto (x + 2\pi i k, y + 2\pi i l)$ changes m by $2\pi i(rk + sl)$. Thus precisely the subgroup K fixes m, so m descends to T_M. The homomorphism

$$\mathbf{Z}^2 \longrightarrow \mathbf{Z}, \qquad (k, l) \longmapsto rk + sl$$

has kernel K and image $d\mathbf{Z}$. The first isomorphism theorem identifies the remaining deck group with $d\mathbf{Z}$, and the chosen Bezout pair gives the displayed generator. Formula (4.14) follows. The logarithm of A changes independently by $2\pi i\mathbf{Z}$, which proves (4.15). □

Lemma 4.18. *Let C be one of the normalized carrier components and let $M|_C$ be proper. For compact sets $L \subset C$ and $K_0 \subset \mathbf{C}$, the part of the pullback*

$$C \times_T T_M$$

lying over L and satisfying $m \in K_0$ is compact. More generally, if m ranges in a compact set, then the underlying carrier point ranges in a compact set; over each fixed underlying point only finitely many character-kernel sheets can have m in that compact set.

Proof. If $m \in K_0$, then $M = e^m$ lies in the compact subset $e^{K_0} \subset \mathbf{C}^*$. Properness of $M|_C$ confines the underlying point to a compact subset of C. Over a fixed point, the possible values of m form the arithmetic progression (4.14), which meets K_0 in a finite set. A finite evenly covered neighborhood argument over a finite cover of the compact base subset now gives compactness of the stated portion of the pullback. □

Lemma 4.19. *Let $p : \widetilde{Z} \to Z$ be an unramified holomorphic covering of complex manifolds.*

(1) If $E \subset Z$ is closed discrete, then the complete inverse image $p^{-1}(E)$ is closed discrete in $\widetilde{Z}$.

(2) The restriction

$$p : \widetilde{Z} \setminus p^{-1}(E) \longrightarrow Z \setminus E$$

is an unramified surjective holomorphic covering.

(3) A biholomorphism sends closed discrete sets to closed discrete sets.

Proof. Every point of Z has an evenly covered neighborhood. If its center is outside E, shrink the neighborhood to avoid E, using closedness. If its center lies in E, shrink it so that it contains no other point of E. Every lifted sheet then meets $p^{-1}(E)$ either in no point or in exactly one point. This proves the first assertion. Removing the full inverse image preserves every sheet over $Z \setminus E$, proving the second. The third assertion follows by applying a homeomorphism and its inverse. □

Lemma 4.20. *Let $n = k + l \geq 2$, where $k, l \geq 1$, and let $E \subset \mathbf{C}^n = \mathbf{C}^k \times \mathbf{C}^l$ be closed discrete. Suppose that the image of E under the coordinate projection to $\mathbf{C}^k$ is closed discrete. Then*

$$\mathbf{C}^n \setminus E$$

is Oka. No finiteness assumption on the projection fibres is required.

Proof. If E is infinite, the Rosay–Rudin projection theorem, Theorem 4.7, makes E tame; its hypothesis concerns the projected image, not the cardinality of the fibres. If E is finite, its projective closure already misses the hyperplane at infinity. In either case E is a zero-dimensional closed analytic subvariety of $\mathbf{C}^n$, so $\dim E \leq n - 2$. The tame-subvariety complement theorem, Theorem 4.8, now gives the conclusion. □

4.3. **Regular suspensions.** Let $T = (\mathbf{C}^*)^2$ and let

$$g(u, v) = \sum_{(i,j)\in\Lambda} c_{ij} u^i v^j \in \mathbf{C}[u^{\pm 1}, v^{\pm 1}] \tag{4.16}$$

be a Laurent polynomial written with distinct exponents and nonzero coefficients. The suspension hypersurface and its regular locus are

$$H_g = \{(q, A, B) \in T \times \mathbf{C}^2 : AB = g(q)\}, \qquad S_g = (H_g)^{\mathrm{reg}}. \tag{4.17}$$

Let $D \subset S_g$ be closed discrete. We impose the following *regular-carrier hypotheses*:

(R1) the base projection of D is contained in a finite union of closed reduced algebraic curves $C_1, \ldots, C_N \subset T$;

(R2) for each ν there is a constant $c_\nu \in \mathbf{C}^*$ such that $g|_{C_\nu} = c_\nu$.

Thus every point of D over C_ν satisfies

$$AB = c_\nu \neq 0. \tag{4.18}$$

Theorem 4.21. *Under hypotheses (R1) and (R2), for every finite set $F \subset S_g$ the manifold*

$$S_g \setminus (D \cup F)$$

is Oka.

The finite set may meet a principal divisor or the smooth axis. No assumption is made on its base values. By replacing F with $F \setminus D$, we may assume in the proof that $D \cap F = \varnothing$.

Write

$$S_A = S_g \cap \{A \neq 0\}, \qquad S_B = S_g \cap \{B \neq 0\}.$$

The first open is exactly

$$S_A \simeq T \times \mathbf{C}^*_A, \qquad B = \frac{g(u,v)}{A},$$

and similarly $S_B \simeq T \times \mathbf{C}^*_B$. There is no further singular-locus deletion on S_A, since $\partial(AB - g)/\partial B = A \neq 0$ there; the corresponding derivative is nonzero on S_B.

Lemma 4.22. *Both*

$$S_A \setminus (D \cup F) \quad \textit{and} \quad S_B \setminus (D \cup F)$$

are Oka.

Proof. Choose a proper character $M = u^r v^s$ by Lemma 4.14. The universal logarithmic covering of the first principal open is

$$\pi_A : \mathbf{C}^3_{x,y,\sigma} \longrightarrow S_A, \qquad (u, v, A, B) = (e^x, e^y, e^\sigma, g(e^x, e^y)e^{-\sigma}). \tag{4.19}$$

Let E_A be the complete inverse image of $(D \cup F) \cap S_A$. By Lemma 4.19, E_A is closed discrete. Consider the rank-two linear projection

$$P_A : \mathbf{C}^3 \longrightarrow \mathbf{C}^2, \qquad P_A(x, y, \sigma) = (m, \sigma) = (rx + sy, \sigma). \tag{4.20}$$

We prove that $P_A(E_A)$ is closed discrete.

First consider the complete lift of D. Suppose that (m, σ) remains in a compact subset of $\mathbf{C}^2$. Then $M = e^m$ and $A = e^\sigma$ remain in compact annuli. Properness of M on the finite carrier family confines the base point (u, v) to a compact subset of the normalized carriers. On C_ν one has $B = c_\nu/A$. Since there are only finitely many nonzero constants c_ν, the coordinate B also remains in a compact annulus. The corresponding points of D therefore lie in a compact subset of S_A. This compact set stays away from the singular locus because both A and B are bounded away from zero. Closed discreteness of D leaves only finitely many downstairs points.

For each such point, the possible projected logarithms are described by Lemma 4.17:

$$(m_0 + 2\pi i d\mathbf{Z}) \times (\sigma_0 + 2\pi i \mathbf{Z}).$$

A compact subset of $\mathbf{C}^2$ meets this product lattice in finitely many points. The kernel branches may be infinite, but they create no new projected value. Hence the projected complete lift of D is closed discrete.

For each point of the finite set $F \cap S_A$, the projected complete lift is one product lattice of the same form. A finite union of these lattices is closed discrete, and adjoining it to the already closed discrete carrier projection preserves closed discreteness. Lemma 4.20 gives

$$\mathbf{C}^3 \setminus E_A \quad \text{Oka.}$$

Because E_A is the complete inverse image, (4.19) restricts to the surjective unramified covering

$$\mathbf{C}^3 \setminus E_A \longrightarrow S_A \setminus (D \cup F).$$

Covering invariance, Theorem 4.1, proves the first assertion. Interchanging A and B proves the second. □

It remains to construct Oka neighborhoods of retained points with $A = B = 0$. Fix such a point

$$e = (q_e, 0, 0) \in S_g \setminus (D \cup F).$$

Then $g(q_e) = 0$, and smoothness of S_g gives $dg_{q_e} \neq 0$. Choose an axis-adapted character by Lemma 4.15. With $u = e^x$, $v = e^y$, use the notation

$$m = rx + sy, \qquad W = su\frac{\partial}{\partial u} - rv\frac{\partial}{\partial v}.$$

The field W is complete on T and has flow

$$\Psi_t(u,v) = (e^{st}u, e^{-rt}v). \tag{4.21}$$

On H_g consider

$$Y = BW + (Wg)\frac{\partial}{\partial A}. \tag{4.22}$$

Lemma 4.23. *The vector field Y is tangent to H_g and complete. Its flow is*

$$\Phi_t(q,A,B) = \big(\Psi_{tB}(q), A + R(t,q,B), B\big), \tag{4.23}$$

where

$$R(t,q,B) = t\int_0^1 (Wg)(\Psi_{\tau t B}(q))\,d\tau. \tag{4.24}$$

For $B \neq 0$ this becomes

$$R(t,q,B) = \frac{g(\Psi_{tB}(q)) - g(q)}{B},$$

while $R(t,q,0) = t(Wg)(q)$. The functions B and M are first integrals. Consequently, for every

$$\alpha \in \mathcal{O}(\mathbf{C}_B \times \mathbf{C}^*_M)$$

the pointwise time map

$$\phi = \Phi_{\alpha(B,M)} \tag{4.25}$$

is an automorphism of H_g and restricts to an automorphism of S_g.

Proof. Tangency follows from

$$Y(AB - g) = B(Wg) - B(Wg) = 0.$$

The integral in (4.24) is jointly holomorphic in all variables and is the removable extension of the displayed difference quotient. Substitution gives

$$(A+R)B = g(q) + g(\Psi_{tB}(q)) - g(q) = g(\Psi_{tB}(q)),$$

so (4.23) preserves H_g. The group law follows from the group law for Ψ and the telescoping identity for the two differences of g. Thus Φ_{-t} is the inverse of Φ_t.

Finally, $YB = 0$ and $YM = BW(M) = 0$. Hence $\alpha(B,M)$ is constant along every Y-orbit, and the inverse of (4.25) is $\Phi_{-\alpha(B,M)}$. Since a biholomorphism preserves the regular locus, the replica restricts to S_g. This is the complete-replica mechanism recorded in Lemma 3.13. □

Split the infinite deletion into

$$D_+ = \{d \in D : |B(d)| \geq 1\}, \qquad D_- = \{d \in D : |B(d)| < 1\},$$

consider

$$\Pi_+ : D_+ \longrightarrow \mathbf{C} \times \mathbf{C}^*, \qquad \Pi_+(d) = (B(d), M(d)). \tag{4.26}$$

Lemma 4.24. *The image $\mathcal{B}_+ = \Pi_+(D_+)$ is closed discrete, and every fibre of Π_+ is finite.*

Proof. Suppose that (B,M) remains in a compact subset of $\mathbf{C}\times\mathbf{C}^*$ along D_+. Then $1 \leq |B| \leq C$. Properness of M confines the base point to a compact part of the finite carrier union. On a carrier C_ν one has $A = c_\nu/B$. Since the set of nonzero constants c_ν is finite, A also remains in a compact annulus. Thus the corresponding points of D lie in a compact subset of S_g, which contains only finitely many of them. This proves that the image is closed discrete.

For fixed (B,M), finiteness of M on each normalized carrier leaves only finitely many possible base points. The equation $A = c_\nu/B$ then fixes A. Hence every fibre is finite. □

Lemma 4.25. *For every $b = (B,M) \in \mathcal{B}_+$ and every $R_0 > 0$, there is $t_b \in \mathbf{C}$ such that*

$$|A(\Phi_{t_b}(d))| > R_0$$

for every $d \in D_+$ satisfying $\Pi_+(d) = b$.

Proof. Write $T_0 = Bt$. For $d = (q, A, B)$ in the fixed finite fibre, (4.23) and the suspension equation give

$$BA_t = g(\Psi_{T_0}(q)) = \sum_{(i,j)\in\Lambda} c_{ij} u^i v^j e^{(is-jr)T_0}. \tag{4.27}$$

The weights $is - jr$ are pairwise distinct. Since $g(q_e) = 0$ at a torus point, g is not a single Laurent monomial, and at least one weight is nonzero. Choose $\epsilon \in \{1, -1\}$ so that the largest number among $\epsilon(is - jr)$ is positive. It is unique. Along $T_0 = \epsilon R$, $R \longrightarrow +\infty$, the associated term in (4.27) dominates. Its coefficient $c_{ij}u^i v^j$ never vanishes. The fibre is finite, so the divergence is uniform on it. Since its common value of B is nonzero, taking R large and setting $t_b = \epsilon R/B$ proves the lemma. □

Choose a proper exhaustion

$$\rho : \mathcal{B}_+ \longrightarrow [1, \infty),$$

meaning that every bounded sublevel is finite. By Lemma 4.25, choose t_b so that

$$|A(\Phi_{t_b}(d))| > e^{\rho(b)} \quad \big(d \in D_+,\ \Pi_+(d) = b\big). \tag{4.28}$$

The set

$$\mathcal{B}_+ \cup \{(0, M(q_e))\} \subset \mathbf{C} \times \mathbf{C}^*$$

is closed discrete because $|B| \geq 1$ on $\mathcal{B}_+$. By the Cartan interpolation theorem, Theorem 4.9, there exists

$$\alpha \in \mathcal{O}(\mathbf{C} \times \mathbf{C}^*), \qquad \alpha(b) = t_b \quad (b \in \mathcal{B}_+), \qquad \alpha(0, M(q_e)) = 1. \tag{4.29}$$

Let $\phi = \Phi_{\alpha(B,M)}$ and set

$$D' = \phi(D), \qquad F' = \phi(F).$$

At the retained axis point,

$$A(\phi(e)) = (Wg)(q_e) \neq 0. \tag{4.30}$$

Proposition 4.26. *The manifold*

$$S_A \setminus (D' \cup F')$$

is Oka.

Proof. Use the logarithmic covering (4.19), with $A = e^{\sigma}$, and let E' be the complete inverse image of $(D' \cup F') \cap S_A$. We prove that the image of E' under

$$(x, y, \sigma) \longmapsto (m, \sigma)$$

is closed discrete. The set E' itself is closed discrete by Lemma 4.19, since ϕ is a biholomorphism and $D \cup F$ is closed discrete.

First suppose that a sequence in D' is the image of a sequence in D_+. If its projected (m, σ)-values remain compact, then $|A'|$ is bounded above. The escape estimate (4.28) bounds $\rho(B, M)$. Only finitely many values of $(B, M) \in \mathcal{B}_+$ can occur, and Lemma 4.24 leaves only finitely many downstairs points. The arithmetic progressions in Lemma 4.17 then leave only finitely many projected logarithmic values.

Now suppose that the preimages lie in D_-. Bounded m confines M to a compact annulus, and properness of M confines q to a compact part of the carrier curves. The set

$$\{|B| \leq 1\} \times \{M \text{ in that compact annulus}\} \subset \mathbf{C} \times \mathbf{C}^*$$

is compact, so $\alpha(B, M)$ is bounded there. Formula (4.24) gives the uniform estimate

$$A' - A = \alpha(B, M) \int_0^1 (Wg)(\Psi_{\tau\alpha(B,M)B}(q))\, d\tau = O(1). \tag{4.31}$$

Indeed, q ranges in a compact set, the flow times $\tau\alpha(B, M)B$ remain bounded, and the image of the resulting compact parameter set under the entire flow Ψ is compact.

If $B \to 0$ along this sequence, then

$$|A| = \frac{|c_\nu|}{|B|} \longrightarrow \infty$$

uniformly over the finite family of nonzero carrier constants. The estimate (4.31) would force $|A'| \to \infty$, contrary to bounded σ. Hence $|B|$ is bounded below. Together with $|B| < 1$, compactness of the base, and $AB = c_\nu$, this confines the original points to a compact subset of S_g. Closed discreteness leaves only finitely many of them, and the logarithmic period calculation again leaves only finitely many projected values.

We have proved that the projected complete lift of D' is closed discrete. The set F' is finite, so the projected complete lift of $F' \cap S_A$ is a finite union of product lattices (4.15). Adjoining it preserves closed discreteness.

Lemma 4.20 makes $\mathbf{C}^3 \setminus E'$ Oka. Since E' is the complete inverse image, the restriction of (4.19) is an unramified surjective covering onto $S_A \setminus (D' \cup F')$. Covering invariance proves the proposition. □

Proof of Theorem 4.21. Set

$$X^\circ = S_g \setminus (D \cup F).$$

Lemma 4.22 gives analytic-Zariski-open Oka neighborhoods of every retained point at which $A \neq 0$ or $B \neq 0$.

For a retained axis point e, carry out the construction above and consider

$$\Omega_e = \phi^{-1}\big(S_A \setminus (D' \cup F')\big). \tag{4.32}$$

It is Oka by Proposition 4.26 and contains e by (4.30). Injectivity of ϕ also shows that $\phi(e) \notin D' \cup F'$. Since $\phi(D \cup F) = D' \cup F'$, one has the exact identity

$$\Omega_e = \phi^{-1}(S_A) \setminus (D \cup F).$$

Its complement relative to the fixed punctured manifold is therefore

$$X^\circ \setminus \Omega_e = X^\circ \cap \phi^{-1}\big(S_g \cap \{A = 0\}\big),$$

which is closed analytic in X°. Hence Ω_e is analytic-Zariski open in X°.

Every point of S_g either lies in one of the two principal opens or is an axis point. Thus the two punctured principal opens and the neighborhoods Ω_e cover X°. Kusakabe localization, Theorem 3.2, proves that X° is Oka. □

We now apply Theorem 4.21 to the regular part of a two-row band. By Theorem 4.11(G3), U_m^{reg} is identified with the regular locus of the conic threefold (4.1) over E_m.

We first record the product neighborhoods needed to scalarize (4.1). The two affine charts of E_m are exchanged by the fibrewise automorphism and have the Danielewski forms (3.6)–(3.7).

Lemma 4.27. *Every point $q \in E_m$ has an algebraic Zariski-open neighborhood $Q \simeq \mathbf{C}^* \times \mathbf{C}$. Moreover, the restriction $\mathcal{L}_m|_Q$ is holomorphically trivial.*

Proof. It suffices to work on a Danielewski surface

$$W_f = \{ab = f(c)\},$$

where f is one of the two polynomials in (3.6)–(3.7); all its roots are simple. The complete algebraic vector fields

$$D_a = a\frac{\partial}{\partial c} + f'(c)\frac{\partial}{\partial b}, \qquad D_b = b\frac{\partial}{\partial c} + f'(c)\frac{\partial}{\partial a}, \qquad H = a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b}$$

span the tangent space everywhere. If $a \neq 0$, use D_a and H; if $b \neq 0$, use D_b and H; and at a point with $a = b = 0$, simplicity of the root gives the independent vectors $f'(c)\partial_b$ and $f'(c)\partial_a$.

Their flows are algebraic automorphisms. For example,

$$\mathrm{Fl}_t^{D_a}(a, b, c) = \left(a,\ b + \frac{f(c + ta) - f(c)}{a},\ c + ta\right),$$

where the displayed quotient is a polynomial. The generated group has open orbits because the infinitesimal directions span. Since W_f is connected, it has a single orbit. Given q, choose an algebraic automorphism ψ carrying q into $\{a \neq 0\}$ and take

$$Q = \psi^{-1}(\{a \neq 0\}).$$

Then Q is algebraic Zariski open, contains q, and is isomorphic to $\mathbf{C}_a^* \times \mathbf{C}_c$.

The Stein manifold Q retracts onto a circle. Hence Lemma 4.3 gives $\mathrm{Pic}(Q) = 0$, and $\mathcal{L}_m|_Q$ is holomorphically trivial. □

Let $\Gamma \simeq \mathbf{Z}$ be the residual deck group. Fix finitely many points in the common principal smooth core, lying over values z_ν such that

$$\Delta_m(z_\nu) \neq 0. \tag{4.33}$$

Let $D \subset U_m^{\mathrm{reg}}$ be the union of their full residual-deck orbits in the band. Proper discontinuity makes D closed discrete. The deck action is vertical over z, so the base projection of each orbit is contained in the algebraic curve

$$C_\nu = \{z = z_\nu\} \subset E_m, \tag{4.34}$$

and

$$\Delta_m|_{C_\nu} = \Delta_m(z_\nu) = c_\nu \in \mathbf{C}^*. \tag{4.35}$$

Theorem 4.28. *For $m = 3, 4$, let $D \subset U_m^{\mathrm{reg}}$ be a finite union of full residual-deck orbits through points of the common principal smooth core, and let $F \subset U_m^{\mathrm{reg}}$ be finite. Then*

$$U_m^{\mathrm{reg}} \setminus (D \cup F)$$

is Oka.

Proof. Set

$$X^\circ = U_m^{\mathrm{reg}} \setminus (D \cup F),$$

fix $x \in X^\circ$, and let $q \in E_m$ be its base point. Choose an algebraic product neighborhood $Q \simeq \mathbf{C}_u^* \times \mathbf{C}_c$ through q by Lemma 4.27. The set D is a finite union of $\mathbf{Z}$-orbits and is therefore countable; adjoining the finite set F preserves countability. Hence the set of c-coordinates of the base projections of $(D \cup F) \cap (U_m^{\mathrm{reg}}|_Q)$ is countable. Choose

$$c_0 \in \mathbf{C}$$

outside this countable set and with $c_0 \neq c(q)$. Then

$$T = Q \setminus \{c = c_0\} \simeq \mathbf{C}_u^* \times \mathbf{C}_v^*, \qquad v = c - c_0, \tag{4.36}$$

is an algebraic Zariski-open neighborhood of q, and the removed level contains no deletion point under consideration.

Choose a holomorphic frame of $\mathcal{L}_m|_Q$ and its dual frame. In these frames the conic equation over T is exactly

$$AB = g(u, v), \qquad g = \Delta_m(z)|_T. \tag{4.37}$$

The function g is a Laurent polynomial. Indeed, in the proof of Lemma 4.27, the coordinates on $Q = \psi^{-1}(\{a \neq 0\})$ are $u = a \circ \psi$ and $c = c_{\mathrm{old}} \circ \psi$. The regular function $z \circ \psi^{-1}$ on $\{a \neq 0\}$ lies in $\mathbf{C}[u, u^{-1}, c]$. Substituting $c = v + c_0$ and then applying the polynomial Δ_m places g in $\mathbf{C}[u^{\pm 1}, v^{\pm 1}]$.

Write

$$D_T = D \cap (U_m^{\mathrm{reg}}|_T), \qquad F_T = F \cap (U_m^{\mathrm{reg}}|_T).$$

The first set is closed discrete. Its base projection lies in the finite family of reduced curves $(C_\nu \cap T)_{\mathrm{red}}$. After splitting these curves into irreducible components, the nonzero constants (4.35) verify hypotheses (R1) and (R2). The second set is finite. Theorem 4.21 therefore gives

$$\Omega_x := (U_m^{\mathrm{reg}}|_T) \setminus (D \cup F) \quad \text{Oka.}$$

This is an analytic-Zariski-open neighborhood of x in the fixed punctured manifold X°. Indeed,

$$\Omega_x = X^\circ \cap (U_m^{\mathrm{reg}}|_T)$$

and its relative complement is the explicit closed analytic subset

$$X^\circ \setminus \Omega_x = X^\circ \cap \big(U_m^{\mathrm{reg}}|_{E_m \setminus T}\big).$$

Every point of X° has such a neighborhood. Kusakabe localization proves that X° is Oka. □

Remark 4.29. The infinite set D in Theorem 4.28 is carried by common-core curves on which Δ_m is a nonzero constant. No infinite orbit over a nodal value is included. The finite set F may contain points over nodal values inside U_m^{reg}. In the global band argument, all remaining nodal and toroidal intersections enter through such a finite set; crossing the exceptional curves requires the corrected incidence-chart theorems of the following sections.

4.4. **Corrected incidence charts.** We now treat the corrected incidence charts along the order-three exceptional curves. The incidence coordinates in this case are scalar coordinates on a product, but the correction deletes a wrong-ruling affine line. The main point is therefore to control the infinite deletion in the uncorrected product chart, while fixing every lift of the wrong line when a replica flow crosses the residual axis.

Use the functions

$$h(w) = 4w^3 + 6w, \qquad q(w) = 1 + w^2. \tag{4.38}$$

Let $w_1, \ldots, w_J \in \mathbb{C}^*$ be distinct. Consider

$$Z_3 = \big(\mathbb{C}_w^* \times \mathbb{C}_A \times \mathbb{C}_p\big) \setminus \bigcup_{j=1}^{J} \ell_j, \qquad \ell_j = \{w = w_j,\ A = 0,\ p \in \mathbb{C}\}. \tag{4.39}$$

Each exact selected chart supplied by Subsection 4.1 has one wrong line. We prove the slightly more robust finite-family statement in (4.39), since the proof is unchanged and the sign-transformed charts are then covered without relabelling the argument.

Write

$$R = Ap. \tag{4.40}$$

Let $D \subset Z_3$ and $F \subset Z_3$ satisfy the following hypotheses.

(H1) The set D is closed and discrete in the uncorrected product $\mathbb{C}^*_w \times \mathbb{C}_A \times \mathbb{C}_p$.

(H2) The set F is finite.

(H3) There are finitely many constants $C_\nu \in \mathbb{C}^*$ such that every point of D satisfies, for at least one ν,

$$C_\nu = R\big(h(w) - q(w)R\big). \tag{4.41}$$

Replacing F by $F \setminus D$ leaves the complement unchanged. We shall therefore assume throughout that

$$D \cap F = \varnothing. \tag{4.42}$$

The nonzero carrier identity implies that R, A, and p do not vanish on D.

The local theorem needed later is the following. Notice that it imposes no invariance or completeness condition on D.

Theorem 4.30. *Under hypotheses (H1)–(H3), the manifold*

$$Z_3 \setminus (D \cup F)$$

is Oka.

We first explain the geometric origin of (4.41). Fix ζ with $\zeta^3 = 1$, using the notation of Theorem 4.11(G5). In the first selected factor chart, the incidence coordinate s and the two factors are

$$s = 2w - R, \qquad L_+ = 2w - s, \qquad G_+ = 2w^3 + 4w + q(w)s.$$

Consequently

$$L_+ = R, \qquad G_+ = 2w^3 + 4w + q(w)(2w - R) = h(w) - q(w)R.$$

The factorization at an order-three point $z = \zeta$ reads

$$z - \zeta = \frac{\zeta}{3} L_+ G_+.$$

On the fixed fibre $z = z_\nu$ this becomes

$$R\big(h(w) - q(w)R\big) = \frac{3(z_\nu - \zeta)}{\zeta} =: C_\nu. \tag{4.43}$$

For a common-principal-core value one has $z_\nu \neq \zeta$, and hence $C_\nu \neq 0$. The calculation takes place in the uncorrected product chart, before the wrong line is removed.

The set D is countable. Indeed, every closed discrete subset of a second-countable space is countable: choose for each point a member of a countable basis which meets the set only at that point. Distinct points can be assigned distinct basis members. Thus $D \cup F$ is countable.

Fix for the moment a number $c \in \mathbb{C}^*$ such that

$$c \notin R(D \cup F). \tag{4.44}$$

On the analytic-Zariski-open set $R \neq c$, use the coordinate

$$u = R - c \in \mathbb{C}^*.$$

The uncorrected incidence space becomes the shifted suspension

$$H_c = \{(w, u, A, p) \in (\mathbb{C}^*)^2 \times \mathbb{C}^2 : Ap = u + c\}, \tag{4.45}$$

and the corrected shifted chart is

$$Z_{3,c} = H_c \setminus \bigcup_{j=1}^{J} \ell_{j,c}, \qquad \ell_{j,c} = \{w = w_j,\ u = -c,\ A = 0,\ p \in \mathbb{C}\}. \tag{4.46}$$

The carrier equation is now

$$(u + c)\big(h(w) - q(w)(u + c)\big) = C_\nu. \tag{4.47}$$

In particular, no carrier meets the level $u = -c$.

The next observation is the precise ambient-closedness check at the wrong lines.

Lemma 4.31. *Suppose that a set satisfying (4.41) is closed discrete in the corrected chart. It has no accumulation point on any wrong line in the uncorrected product chart. In particular, under hypothesis (H1), the union of $D \cup F$ with the wrong lines is a closed analytic subset of the uncorrected chart, and its zero-dimensional part is disjoint from the lines.*

Proof. Every wrong line has $R = Ap = 0$, whereas (4.41) and $C_\nu \neq 0$ give $R \neq 0$ on D. Suppose that a sequence in D converges to a point of a wrong line. Passing to a subsequence, we may keep the carrier index ν fixed. Along the sequence w remains bounded and $R \to 0$. Hence

$$R\big(h(w) - q(w)R\big) \longrightarrow 0,$$

contrary to its constant nonzero value C_ν. This excludes every new accumulation point on the deleted lines. The finite set $F \subset Z_3$ is disjoint from those lines by definition. A locally finite union of affine lines is analytic, and a closed discrete subset is a zero-dimensional analytic subvariety; their disjoint union is therefore closed analytic. □

Let $\Gamma_\nu \subset (\mathbb{C}^*)^2_{w,u}$ be the reduced algebraic curve defined by (4.47), and retain only its irreducible components which contain a base projection of D. By Lemma 4.16, there is an integer b such that

$$M(w,u) = wu^b \tag{4.48}$$

is proper and finite on the normalization of every such component. If a retained axis point over $(w_e, -c)$ has been fixed, we include w_e among the finitely many w-coordinates in that lemma. Since $w_e \neq w_j$, this choice also gives

$$M(w_e, -c) \neq M(w_j, -c) \quad (1 \le j \le J). \tag{4.49}$$

Choose logarithms

$$w = e^x, \qquad u = e^y,$$

and set

$$m = x + by, \qquad n = -bx + y. \tag{4.50}$$

The kernel of the character on the logarithmic deck lattice is

$$K = \{(k,l) \in \mathbb{Z}^2 : k + bl = 0\} = \mathbb{Z}(b,-1). \tag{4.51}$$

The character-kernel intermediate cover is

$$B_K = \mathbb{C}^2_{x,y}/(2\pi i K) \simeq \mathbb{C} \times \mathbb{C}^*, \qquad [x,y] \longmapsto \left(m, \exp\left(-\frac{n}{1+b^2}\right)\right). \tag{4.52}$$

This is the specialization of Lemma 4.17 to the primitive character $(1,b)$. A generator of K changes n by

$$n \longmapsto n - 2\pi i(1+b^2), \tag{4.53}$$

while fixing m. Thus every fibre left after passage to B_K becomes on the cyclic cover a finite union of progressions parallel to the imaginary n-axis. Infinite progression fibres will be retained; the projection criterion used below does not require finite fibres.

The universal base-logarithmic pullback of (4.45) is

$$\mathcal{H}_c = \{(x,y,A,p) \in \mathbb{C}^4 : Ap = f(y)\}, \qquad f(y) = e^y + c. \tag{4.54}$$

It is smooth because $f_y = e^y$ never vanishes. Let $\widetilde{D}$, $\widetilde{F}$, and L denote the complete inverse images in $\mathcal{H}_c$ of D, F, and $\bigcup_j \ell_{j,c}$, respectively. The word "complete" refers to the covering inverse image; no invariance condition on D is involved. By Lemma 4.19 and Lemma 4.31,

$$\widetilde{D} \cup \widetilde{F} \cup L \tag{4.55}$$

is a closed analytic subset of $\mathcal{H}_c$, with discrete part disjoint from the locally finite union of lines L.

We shall repeatedly use compact lifting modulo K. Namely, if m stays in a compact set on a lifted carrier, then $M = e^m$ stays in a compact annulus. Properness of M confines the downstairs carrier point to a compact set, and Lemma 4.18 confines its lift to a compact subset of the intermediate cover.

We first treat points away from the residual axis.

Proposition 4.32. *The manifold*

$$\mathcal{H}_c \cap \{A \neq 0\} \setminus (\widetilde{D} \cup \widetilde{F})$$

is Oka.

Proof. Write $A = e^{\sigma}$. Eliminating p gives the unramified covering

$$\pi_A : \mathbb{C}^3_{x,y,\sigma} \longrightarrow \mathcal{H}_c \cap \{A \neq 0\}, \qquad p = (e^y + c)e^{-\sigma}. \tag{4.56}$$

Let E_A be the complete inverse image of $(\widetilde{D} \cup \widetilde{F}) \cap \{A \neq 0\}$. It is closed discrete by Lemma 4.19. In the linear coordinates (σ, m, n), consider

$$P_A(\sigma, m, n) = (\sigma, m).$$

We claim that $P_A(E_A)$ is closed discrete.

Let (σ, m) remain in a compact set on the part over D. Then $A = e^{\sigma}$ and $M = e^m$ remain in compact annuli. Properness of M confines the downstairs base to a compact subset of the finite carrier union, and compact lifting modulo K confines the intermediate base lift. The formula

$$p = \frac{u + c}{A}$$

bounds the remaining fibre coordinate. The complete lift of D to the intermediate incidence space is closed discrete, so only finitely many intermediate points occur. For each such point, bounded σ leaves only finitely many logarithms of A. The remaining K-lifts change only n by (4.53) and create no new projected value.

For a point of the finite set F, the possible m-values of all base lifts form one translate of $2\pi i\mathbb{Z}$. A compact m-set meets it finitely, and bounded σ leaves finitely many logarithms of A. The residual K-lifts again change only n. A finite union over F is therefore locally finite in the (σ, m)-plane. This proves the claim.

Lemma 4.20 makes $\mathbb{C}^3 \setminus E_A$ Oka. Since E_A is the complete inverse image, (4.56) restricts to a surjective unramified covering of the displayed punctured principal open. Covering invariance, Theorem 4.1, proves the proposition. No wrong line occurs on this open because every wrong line has $A = 0$. □

Proposition 4.33. *The manifold*

$$\mathcal{H}_c \cap \{p \neq 0\} \setminus (\widetilde{D} \cup \widetilde{F} \cup L)$$

is Oka.

Proof. Write $p = e^{\rho}$. Eliminating A gives an unramified covering by $\mathbb{C}^3_{x,y,\rho}$, with

$$A = (e^y + c)e^{-\rho}. \tag{4.57}$$

Let E_p be the complete inverse image of $\widetilde{D} \cup \widetilde{F}$, and let $\widehat{L}$ be the complete inverse image of L. Choose logarithms $e^{x_j} = w_j$ and $e^{y_c} = -c$. Then

$$\widehat{L} = \bigcup_{j=1}^{J} \bigcup_{k,l \in \mathbb{Z}} \{x = x_j + 2\pi i k,\ y = y_c + 2\pi i l,\ \rho \in \mathbb{C}\}. \tag{4.58}$$

This union is locally finite. Ambient closedness and disjointness give a closed analytic subvariety

$$\Sigma = E_p \cup \widehat{L} \subset \mathbb{C}^3, \qquad \dim \Sigma \leq 1. \tag{4.59}$$

Consider the projection

$$P_p(\rho, m, n) = (\rho, m).$$

Its image

$$B = P_p(E_p) \subset \mathbb{C}^2_{\rho,m}$$

is closed discrete. Indeed, bounded (ρ, m) places $p = e^{\rho}$ and $M = e^m$ in compact annuli. Properness of M and compact lifting modulo K confine the intermediate base over D, while $A = (u + c)/p$ is bounded. Closed discreteness on the intermediate cover leaves finitely many points. For F, bounded m allows only finitely many sheets modulo K, and bounded ρ allows only finitely many logarithms of p. For each fixed point of B, the remaining n-values form a finite union of progressions with period $-2\pi i(1 + b^2)$; their real parts therefore form a finite set.

The m-values on the lifted wrong lines form the closed discrete set

$$S_L = \bigcup_{j=1}^{J} (m_j + 2\pi i\mathbb{Z}), \qquad m_j = x_j + b y_c. \tag{4.60}$$

For fixed $s \in S_L$, the wrong-line n-values are a finite union of progressions in the direction (4.53); in particular, their real parts form a finite set. Since B is closed discrete, the set

$$B^0_s = \{(\rho, s) \in B : |\rho| \leq 2(|s| + 1)\}$$

is finite. Choose a positive real number c_s so large that

$$|\operatorname{Re} n + c_s| \geq 2(|s| + 1) \tag{4.61}$$

for every real part occurring either on a lifted wrong line at level s or in a deletion fibre over B_s^0. By Theorem 4.9, there is an entire function φ with $\varphi(s) = c_s$ on S_L. By Theorem 4.10, choose an entire function γ whose zero set is exactly S_L.

The subset

$$B' = \{(\rho, m) \in B : m \notin S_L\}$$

is closed discrete. For each $q = (\rho, m) \in B'$, choose $c_q > 0$ such that

$$|\operatorname{Re} n + c_q| \geq 2(|m| + 1) \tag{4.62}$$

for every n in the fibre over q. This is possible because only finitely many real parts occur. Cartan interpolation on B' gives $G \in \mathcal{O}(\mathbb{C}^2)$ satisfying

$$G(\rho, m) = \frac{c_q - \varphi(m)}{\gamma(m)} \quad \text{at } q = (\rho, m) \in B'.$$

The triangular automorphism fixes ρ, m and sends n to

$$n' = n + \varphi(m) + \gamma(m) G(\rho, m). \tag{4.63}$$

We verify the required projective estimate without suppressing any case. Every point of Σ belongs to exactly one of the following four classes:

(a) it lies on a lifted wrong line;
(b) it lies in E_p, and $m \notin S_L$;
(c) it lies in E_p, one has $m \in S_L$, and $|\rho| \leq 2(|m| + 1)$;
(d) it lies in E_p, one has $m \in S_L$, and $|\rho| > 2(|m| + 1)$.

In cases (a) and (c), equation (4.61) gives $|n'| \geq 2(|m| + 1)$. In case (b), the same bound follows from (4.62). In case (d), one has $|m| < |\rho|/2$. Hence the sheared analytic set is contained in

$$\{(\rho, n', m) \in \mathbb{C}^3 : |m| \leq 1 + \|(\rho, n')\|\}. \tag{4.64}$$

Its projective closure misses the pure m-direction at infinity, and therefore cannot contain the hyperplane at infinity. Thus Σ is a tame analytic subvariety. Since $\dim \Sigma \leq 1 = 3 - 2$, Theorem 4.8 makes $\mathbb{C}^3 \setminus \Sigma$ Oka. The deletion in this Euclidean cover is the complete inverse image of $\widetilde{D} \cup \widetilde{F} \cup L$ on the p-principal open. Covering invariance, Theorem 4.1, proves the proposition. □

It remains to treat points with $A = p = 0$. Fix a retained point

$$e = (x_e, y_e, 0, 0) \in \mathcal{H}_c \setminus (\widetilde{D} \cup \widetilde{F} \cup L). \tag{4.65}$$

Then $e^{y_e} = -c$, and $e^{x_e} = w_e$ differs from every w_j. Choose b as in (4.48), now including w_e in the separation condition. All preceding projection arguments hold for this choice.

Set

$$W = b\partial_x - \partial_y, \qquad Y = pW + (Wf)\partial_A = p(b\partial_x - \partial_y) - e^y \partial_A. \tag{4.66}$$

Lemma 4.34. *The vector field Y is tangent to $\mathcal{H}_c$ and complete. The functions p and $m = x + by$ are first integrals. For every $\alpha \in \mathcal{O}(\mathbb{C}^2_{p,m})$, the field $\alpha(p, m)Y$ is complete, and its time-one map Φ_α satisfies*

$$x' = x + bT, \qquad y' = y - T, \qquad p' = p, \qquad pA' = e^y e^{-T} + c, \qquad T = \alpha(p, m)p. \tag{4.67}$$

At $p = 0$, the last expression is understood through its holomorphic difference-quotient extension. Moreover,

$$Y(e) = c\partial_A \neq 0.$$

Proof. For $F_0 = Ap - f(y)$ one has

$$W(F_0) = -Wf, \qquad \partial_A(F_0) = p,$$

and hence $Y(F_0) = 0$. Also $Yp = 0$, while $Ym = pWm = 0$. Along a trajectory, p and m are constant, so the base variables translate linearly. For $p \neq 0$, the incidence equation gives the last formula in (4.67); at $p = 0$ one may write

$$A' = A + \alpha(p, m) \int_0^1 (Wf)(x + tbT, y - tT)\, dt,$$

which is holomorphic in all variables. The flow is therefore defined for every complex time. Since $\alpha(p,m)$ is a first integral, the complete replica lemma, Lemma 3.13, makes αY complete. Finally, $e^{y_e} = -c$ gives $Wf(e) = -e^{y_e} = c$. □

Let D_K be the complete inverse image of D in the incidence space over B_K, and split it into

$$D_+ = D_K \cap \{|p| \geq 1\}, \qquad D_- = D_K \cap \{|p| < 1\}.$$

Lemma 4.35. *The image of D_+ under*

$$\Pi_p = (p, m)$$

is closed discrete. The map

$$\Pi_A = (A, m) : D_- \longrightarrow \mathbb{C}^2$$

is proper onto a closed discrete image. On the cyclic cover, every fibre of either projection is a finite union of progressions in the direction (4.53).

Proof. Suppose first that (p, m) remains in a compact set on D_+. Since $|p| \geq 1$, the coordinate p lies in a compact annulus. Properness of M confines the downstairs carrier base, and compact lifting modulo K confines its intermediate lift. The identity $A = (u + c)/p$ then bounds A. The complete lift D_K is closed discrete, so only finitely many points occur. This proves that $\Pi_p(D_+)$ is closed discrete.

Now suppose that (A, m) remains in a compact set on D_-. Properness again confines the base to a compact carrier set, and compact lifting modulo K confines the intermediate base lift. On this set $u + c = R$ is nonzero by (4.41) and therefore has a positive lower bound. Since $Ap = u + c$ and A is bounded, p is bounded away from zero; by definition $|p| < 1$. All coordinates consequently lie in a compact subset of the intermediate incidence space, and closed discreteness gives finiteness. Thus Π_A is proper and its image is closed discrete.

For a fixed projected value, properness and finiteness of M leave only finitely many intermediate points. Their remaining universal lifts are the K-progressions (4.53). □

Write

$$B_+ = \Pi_p(D_+), \qquad B_+^\circ = \{(p, m) \in B_+ : m \notin S_L\}.$$

The second set is closed discrete. For a point $q = (p, m) \in B_+^\circ$, Lemma 4.35 leaves only finitely many representatives modulo K. For all of them, formula (4.67) gives

$$A' = \frac{ue^{-\alpha(q)p} + c}{p}. \tag{4.68}$$

Here $u \neq 0$. Taking $\alpha(q) = -R/p$ with $R \to +\infty$ makes ue^R dominate c for all finitely many representatives at once. Choose a proper exhaustion $\eta : B_+^\circ \to [0, \infty)$ and choose a time t_q so that

$$|A'| > e^{\eta(q)} \quad \text{for every representative over } q. \tag{4.69}$$

The separation (4.49) says precisely that

$$m_e = x_e + by_e \notin S_L. \tag{4.70}$$

Choose a nonzero number t_e. Since $|p| \geq 1$ on B_+°, the set

$$B_+^\circ \cup \{(0, m_e)\}$$

is closed discrete. Cartan interpolation gives $G_e \in \mathcal{O}(\mathbb{C}^2)$ such that

$$G_e(q) = \frac{t_q}{\gamma(m_q)} \quad (q = (p_q, m_q) \in B_+^\circ), \qquad G_e(0, m_e) = \frac{t_e}{\gamma(m_e)}.$$

Set

$$\alpha(p, m) = \gamma(m) G_e(p, m), \qquad \Phi = \Phi_\alpha. \tag{4.71}$$

The coefficient vanishes on S_L, so Φ fixes every component of L pointwise. At e, one has $\alpha(0, m_e) = t_e \neq 0$ and therefore

$$A(\Phi(e)) = ct_e \neq 0. \tag{4.72}$$

The vector field and its coefficient are K-invariant; hence Φ commutes with K and acts on the intermediate incidence cover. It need not commute with the full logarithmic deck group, and no such descent will be used.

Proposition 4.36. *Write*

$$D' = \Phi(\widetilde{D}), \qquad F' = \Phi(\widetilde{F}).$$

On the logarithmic cover of $\mathcal{H}_c \cap \{A \neq 0\}$, the complete inverse image of $(D' \cup F') \cap \{A \neq 0\}$ is closed discrete and has closed discrete (σ, m)-projection. Consequently

$$\mathcal{H}_c \cap \{A \neq 0\} \setminus (D' \cup F') \quad \textit{is Oka.} \tag{4.73}$$

Proof. Automorphisms preserve closed discreteness. We prove local finiteness of the projected complete lift. Equivalently, let m remain in a compact set and let the transformed nonzero coordinate $A' = e^{\sigma}$ remain in a compact annulus.

For points originating in D_+ with $m \notin S_L$, the escape estimate (4.69) bounds $\eta(p, m)$. Properness of the exhaustion leaves only finitely many points of B_+°, and Lemma 4.35 leaves only finitely many intermediate representatives. For points in D_+ with $m \in S_L$, the replica time is zero. Properness of M confines the base to a compact carrier set, and compact lifting modulo K confines the intermediate base lift. The compact-annulus bound on $A' = A$ bounds $p = (u + c)/A$. Together with $|p| \geq 1$, this confines the original points to a compact subset of the intermediate incidence space.

Consider points originating in D_-. Bounded m again confines the base to a compact carrier set. If $|p|$ is bounded below, the identity $A = (u + c)/p$ bounds the original A. Compact lifting modulo K also confines the intermediate base lift, so closed discreteness gives finiteness. It remains to exclude a sequence with $p \to 0$. On $|p| \leq 1$ with m in a compact set, the entire function $\alpha(p, m)$ is bounded. Formula (4.67) gives

$$A' - A = e^{y} \frac{e^{-\alpha(p,m)p} - 1}{p}. \tag{4.74}$$

The right-hand side is uniformly bounded on the compact carrier set by the Taylor expansion of the exponential. On the same set $|u + c|$ has a positive lower bound. Hence

$$|A| = \frac{|u + c|}{|p|} \longrightarrow \infty,$$

and (4.74) forces $|A'| \to \infty$, contradicting the compact-annulus bound on A'. The transformed lift of D therefore has closed discrete (σ, m)-projection.

For the finite set F, bounded m permits only finitely many intermediate base lifts. The K-equivariant automorphism preserves m and sends each of these lifts to one intermediate point. Logging a nonzero A' and bounding σ leaves only finitely many logarithm branches; the remaining K-branches change only n. Thus the transformed complete lift of F also has closed discrete (σ, m)-projection. A finite union of the two projected sets is closed discrete.

Lemma 4.20 makes the Euclidean logarithmic complement Oka. The logarithmic cover removes the complete inverse image of $(D' \cup F') \cap \{A \neq 0\}$, so covering invariance, Theorem 4.1, gives (4.73). □

The required neighbourhood is

$$\Omega_e = \Phi^{-1}\left(\mathcal{H}_c \cap \{A \neq 0\} \setminus (D' \cup F')\right). \tag{4.75}$$

This set is Oka and contains e by (4.72). It omits $\widetilde{D} \cup \widetilde{F}$ because $D' \cup F'$ is their exact image. It also omits L: the automorphism fixes L pointwise and $L \subset \{A = 0\}$. In the fixed punctured corrected manifold

$$\mathcal{X}_c = \mathcal{H}_c \setminus (\widetilde{D} \cup \widetilde{F} \cup L),$$

one has the exact identity

$$\Omega_e = \mathcal{X}_c \cap \{A \circ \Phi \neq 0\}. \tag{4.76}$$

Hence Ω_e is analytic-Zariski open in $\mathcal{X}_c$.

Theorem 4.37. *For every $c \in \mathbb{C}^*$ satisfying (4.44),*

$$Z_{3,c} \setminus (D \cup F)$$

is Oka.

Proof. Work first on $\mathcal{X}_c$. Its intersection with $\{A \neq 0\}$ is Oka by Proposition 4.32, and its intersection with $\{p \neq 0\}$ is Oka by Proposition 4.33. Both are analytic-Zariski open in $\mathcal{X}_c$. Every point not covered by these two sets lies on the residual axis $\{A = p = 0\}$. For each retained axis point, (4.75)–(4.76) give an analytic-Zariski-open Oka neighbourhood. Theorem 3.2 therefore makes $\mathcal{X}_c$ Oka.

The projection

$$\pi : \mathcal{H}_c \longrightarrow H_c, \qquad (x, y, A, p) \longmapsto (e^x, e^y, A, p), \tag{4.77}$$

is the base change of the universal exponential cover of $(\mathbb{C}^*)^2$. It is a surjective unramified holomorphic covering, and by definition

$$\widetilde{D} = \pi^{-1}(D), \qquad \widetilde{F} = \pi^{-1}(F), \qquad L = \pi^{-1}\left(\bigcup_j \ell_{j,c}\right).$$

Thus restriction gives the exact covering

$$\mathcal{H}_c \setminus (\widetilde{D} \cup \widetilde{F} \cup L) \longrightarrow Z_{3,c} \setminus (D \cup F). \tag{4.78}$$

Covering invariance, Theorem 4.1, descends the Oka property and proves the theorem. □

Proof of Theorem 4.30. Write

$$X_3 = Z_3 \setminus (D \cup F)$$

and fix $z \in X_3$. Countability of $D \cup F$ permits the choice

$$c \in \mathbb{C}^* \setminus \big(R(D \cup F) \cup \{R(z)\}\big). \tag{4.79}$$

Then

$$U_c = X_3 \cap \{R \neq c\}$$

contains z. Its relative complement is the closed analytic hypersurface $X_3 \cap \{R = c\}$, so U_c is analytic-Zariski open in X_3. Condition (4.79) also ensures that every point of $D \cup F$ belongs to the shifted open. The substitution $u = R - c$ therefore identifies U_c with the exact complete deletion in Theorem 4.37; hence U_c is Oka. Every point of X_3 has such a neighbourhood, and one final application of Theorem 3.2 proves the result. □

We finish by checking the companion and sign-transformed charts. In the companion selected chart, the scalar incidence coordinate is $R_B = Bp_B$ and

$$s = -2w + R_B, \qquad L_- = 2w + s, \qquad G_- = 2w^3 + 4w - q(w)s.$$

Direct substitution gives

$$L_- = R_B, \qquad G_- = h(w) - q(w)R_B.$$

Thus the companion factorization gives exactly

$$C_\nu = R_B\big(h(w) - q(w)R_B\big), \tag{4.80}$$

and relabelling (B, p_B) as (A, p) applies Theorem 4.30 without any coordinate change between the two selected charts.

For the two sign transforms, the exact homogeneous-coordinate identities are

$$\tau^* u_- = -v_+, \qquad \tau^* v_- = -u_+, \qquad [\xi_- : \eta_-] = [\eta_+ : -\xi_+]. \tag{4.81}$$

On the plus A-chart the coordinate is $p_A = \eta_+/\xi_+$, and on the minus B-chart it is $p_B = \xi_-/\eta_-$. The last identity in (4.81) gives $p_B \circ \tau = -p_A$. Since $v_+ = Ap_A$ and $u_- = Bp_B$, the first identity gives

$$(B \circ \tau)(-p_A) = -Ap_A.$$

Hence $B \circ \tau = A$ where $p_A \neq 0$. Both sides are regular functions on the product chart, so the identity extends across $p_A = 0$. The affine sign maps on the uncorrected product charts are therefore

$$\tau_{A\to B}(w, A, p_A) = (-w, A, -p_A), \tag{4.82}$$

$$\tau_{B\to A}(w, B, p_B) = (-w, B, -p_B). \tag{4.83}$$

They are inverse biholomorphisms and carry every wrong line to its corresponding wrong line. In particular, they preserve ambient closed discreteness and finiteness. If $R' = Bp_B = -Ap_A = -R$, then $h(-w) = -h(w)$ and $q(-w) = q(w)$, so

$$R'\big(h(-w) - q(-w)R'\big) = R\big(h(w) - q(w)R\big).$$

Thus the nonzero carrier identity is preserved as well.

Remark 4.38. The infinite deletion in Theorem 4.30 may be any ambient-closed discrete subset contained in finitely many carriers (4.41) with nonzero constants. The theorem does not apply to an arbitrary closed discrete subset of the chart, nor to an infinite zero-carrier deletion. The finite set F is unrestricted inside the corrected chart.

This section proves the local theorem needed near the order-four exceptional curves. Two features distinguish this case from an ordinary scalar suspension. First, the two incidence coordinates take values in a line bundle and its dual over a two-dimensional algebraic torus; this line bundle is not assumed to be trivial. Second, after the quadratic normalization, a selected incidence chart contains one affine exceptional line whose ruling disagrees with the chosen small resolution. We shall trivialize the incidence line only on a character-adapted intermediate cover, and every automorphism used below will fix the full inverse image of the wrong-ruling line.

Fix a fourth root of unity ζ, and set

$$a_0 = i\zeta, \qquad H(h) = 2h^3 + 3\zeta^2 h, \qquad q(h) = a_0 - H(h). \tag{4.84}$$

Let

$$T = (\mathbb{C}^*)^2_{h,a},$$

let $\mathcal{L} \to T$ be the incidence line bundle, and denote its dual by $\mathcal{L}^\vee$. The intrinsic incidence threefold is

$$Z = \left\{(h, a, A, p) : A \in \mathcal{L}_{(h,a)}, \quad p \in \mathcal{L}^\vee_{(h,a)}, \quad \langle A, p\rangle = a - q(h)\right\}. \tag{4.85}$$

It is smooth. Indeed, in any local pair of dual frames its equation is $Ap - a + q(h) = 0$, whose derivative with respect to a is -1.

The exact selected chart has one wrong-ruling line. Write its base point as

$$t_w = (h_w, q(h_w)) \in T, \qquad q(h_w) \neq 0,$$

and set

$$\ell = \{(t_w, A, p) \in Z : A = 0\}. \tag{4.86}$$

Thus $\ell \simeq \mathbb{C}$, with p as its affine coordinate, and the corrected selected chart is

$$U_4 = Z \setminus \ell.$$

Let $D_0 \subset U_4$ be a closed discrete set satisfying the following regular-carrier hypotheses.

(H1) The set D_0 is closed and discrete in the uncorrected incidence threefold Z, not merely in U_4.

(H2) The base image of D_0 is contained in a finite union of affine algebraic curves

$$C_\nu = \left\{(h, a) \in (\mathbb{C}^*)^2 : (a - a_0)^2 - H(h)^2 = 2ia(z_\nu - \zeta)\right\}, \qquad z_\nu^4 \neq 1. \tag{4.87}$$

(H3) On C_ν set

$$N_+ = a - a_0 + H(h), \qquad N_- = a - a_0 - H(h).$$

Then

$$N_+N_- = 2ia(z_\nu - \zeta), \tag{4.88}$$

and the selected incidence chart satisfies

$$\langle A, p\rangle = N_+ \quad \text{on } D_0. \tag{4.89}$$

Since $a \neq 0$ on T and $z_\nu^4 \neq 1$ implies $z_\nu \neq \zeta$, equation (4.88) shows that neither factor vanishes on a carrier. In particular, every point of D_0 has $A \neq 0$ and $p \neq 0$. On a compact subset of the finite carrier union, $|N_+|$ has a positive lower bound: the numerator $|2ia(z_\nu - \zeta)|$ is bounded below and $|N_-|$ is bounded above.

Let $F_0 \subset U_4$ be finite. Replacing F_0 by $F_0 \setminus D_0$ does not change the complement, so we henceforth assume

$$D_0 \cap F_0 = \varnothing. \tag{4.90}$$

Our local result is the following.

Theorem 4.39. *Under hypotheses (H1)–(H3), the corrected incidence chart*

$$U_4 \setminus (D_0 \cup F_0)$$

is Oka.

The first lemma selects a monomial character. Its additional clauses will be used only when we cross the residual axis.

Lemma 4.40. *There are integers r, s, not both zero, such that the character*

$$M(h, a) = h^r a^s$$

is proper and finite on the normalization of every irreducible component of the curves in (4.87). Moreover, if

$$e = (x_e, y_e, 0, 0)$$

is a retained point over the residual axis on the universal exponential cover, then r, s may be chosen so that, with

$$m = rx + sy, \qquad W = s\partial_x - r\partial_y,$$

the following additional properties hold:

(i) $(Wf)(e) \neq 0$*, where*

$$f(x, y) = e^y - q(e^x);$$

(ii) $M(e^{x_e}, e^{y_e}) \neq M(h_w, q(h_w))$*;*

(iii) *the four integers $-r, 0, s, 3s$ are pairwise distinct.*

Proof. Let $\overline{C}$ be the smooth projective normalization of the projective closure of an irreducible carrier component. At a boundary point $b \in \overline{C} \setminus C$, the valuation vector

$$\big(v_h(b), v_a(b)\big) = \big(\mathrm{ord}_b\, h, \mathrm{ord}_b\, a\big)$$

is nonzero. Otherwise both h and a would extend to b with finite nonzero values, contradicting that b lies over the boundary of $C \subset (\mathbb{C}^*)^2$. There are only finitely many such vectors. If

$$r v_h(b) + s v_a(b) \neq 0 \tag{4.91}$$

at every boundary point of every component, then M has a zero or a pole at every boundary point. Hence the inverse image of a compact annulus in $\mathbb{C}^*$ is compact in C. The restriction of M is therefore proper. It is nonconstant by (4.91), and consequently finite.

It remains to incorporate the three conditions at e. Since

$$Wf = -s\, q'(e^x)e^x - re^y,$$

the equation $(Wf)(e) = 0$ excludes at most one rank-one subgroup of $\mathbb{Z}^2$. Pairwise distinctness of $-r, 0, s, 3s$ excludes the four rank-one subgroups defined by

$$r = 0, \qquad s = 0, \qquad r + s = 0, \qquad r + 3s = 0.$$

Finally, set

$$\lambda = \frac{e^{x_e}}{h_w}, \qquad \mu = \frac{e^{y_e}}{q(h_w)}.$$

The pair (λ, μ) is not $(1, 1)$, since otherwise e would lie on the lifted wrong line. The forbidden characters form the proper subgroup

$$H_0 = \{(r, s) \in \mathbb{Z}^2 : \lambda^r \mu^s = 1\}.$$

Together with (4.91) and the preceding conditions, we have excluded H_0 and finitely many subgroups of rank at most one.

This finite union does not cover $\mathbb{Z}^2$. If H_0 has rank at most one, choose an integral vector outside the finitely many rational lines spanned by all the excluded subgroups. If H_0 has rank two, choose $v \notin H_0$ and choose $u \in H_0$ outside the rational lines spanned by the other excluded subgroups. The affine lattice line $v + \mathbb{Z}u$ is disjoint from H_0 and meets each remaining excluded subgroup in at most one point. Some point of this affine line is therefore admissible. This proves all assertions. □

Fix an admissible pair (r, s), set

$$m = rx + sy, \qquad n = -sx + ry, \qquad d = \gcd(|r|, |s|), \tag{4.92}$$

and let

$$K = \{(k, l) \in \mathbb{Z}^2 : rk + sl = 0\} = \mathbb{Z}\left(\frac{s}{d}, -\frac{r}{d}\right). \tag{4.93}$$

Lemma 4.41. *The quotient*

$$B_K = \mathbb{C}^2_{x,y}/(2\pi i K)$$

is an intermediate unramified cover of T and is biholomorphic to $\mathbb{C}\times\mathbb{C}^*$. The pullback of $\mathcal{L}$ to B_K is holomorphically trivial. Consequently, on the universal exponential cover one may choose dual scalar fibre coordinates A, p which are fixed by every K-translation and satisfy

$$Ap = f(x,y) = e^y - q(e^x). \tag{4.94}$$

Proof. The generator $(s/d, -r/d)$ of K fixes m and changes n by $-2\pi i(r^2+s^2)/d$. Hence

$$(x,y) \longmapsto \left(m, \exp\left(-\frac{d}{r^2+s^2}n\right)\right) \tag{4.95}$$

induces a biholomorphism $B_K \simeq \mathbb{C}\times\mathbb{C}^*$. This Stein manifold retracts onto a circle, so Lemma 4.3 gives $\operatorname{Pic}(B_K) = 0$. Thus the pulled-back incidence line has a nowhere-vanishing holomorphic frame. Pulling this frame and its dual to $\mathbb{C}^2$ gives scalar coordinates fixed by K. Their pairing is intrinsic, so the incidence equation becomes exactly (4.94). ◻

This is the only trivialization used in the proof. In particular, no triviality of $\mathcal{L}$ on T is assumed.

Let

$$Z_K = Z \times_T B_K$$

be the incidence threefold over the character-kernel cover, and let $D_K \subset Z_K$ be the complete inverse image of D_0 under the natural covering $Z_K \to Z$. The set D_K is closed and discrete. Indeed, the covering is a local biholomorphism; around any point of Z_K, choose an evenly covered neighbourhood whose image meets the closed discrete set D_0 in at most one point.

We shall need the following compactness consequence.

Lemma 4.42. *Let $C \subset T$ be compact and $R > 0$. The set of points of B_K which map into C and satisfy $|m| \le R$ is compact.*

Proof. Compactness of $C \subset (\mathbb{C}^*)^2$ bounds $\operatorname{Re} x = \log|h|$ and $\operatorname{Re} y = \log|a|$. Hence it bounds $\operatorname{Re} n = -s\operatorname{Re} x + r\operatorname{Re} y$, so the second coordinate in (4.95) remains in a compact annulus. The first coordinate remains in the closed disc $|m| \le R$. The set under consideration is closed in the product of these two compact sets. ◻

Let

$$\epsilon : \mathbb{C}^2_{x,y} \longrightarrow T, \qquad \epsilon(x,y) = (e^x, e^y)$$

be the universal exponential covering. In the frame from Lemma 4.41, its incidence pullback is

$$\mathcal{H} = \{(x,y,A,p) \in \mathbb{C}^4 : Ap = f(x,y)\}. \tag{4.96}$$

It is smooth because $f_y = e^y \neq 0$. Let $D, E, L \subset \mathcal{H}$ be the complete inverse images of D_0, F_0, ℓ, respectively. Choose logarithms $e^{\alpha_w} = h_w$ and $e^{\beta_w} = q(h_w)$. Then

$$L = \bigcup_{k,l\in\mathbb{Z}} \{x = \alpha_w + 2\pi i k,\ y = \beta_w + 2\pi i l,\ A = 0,\ p \in \mathbb{C}\}. \tag{4.97}$$

The set $D \cup E$ is closed discrete, while L is a locally finite closed analytic union of lines. The three sets are pairwise disjoint by (4.90) and the inclusions $D_0, F_0 \subset U_4$.

The next lemma records both the regular-carrier and finite parts of the deletion. It will be used in both principal opens and again after the replica automorphism.

Lemma 4.43. *Use the frame and coordinates fixed above.*

(a) *On $\mathcal{H}_A = \mathcal{H} \cap \{A \neq 0\}$, write $A = e^\sigma$. Let $\widehat{D}_A, \widehat{E}_A \subset \mathbb{C}^3_{x,y,\sigma}$ be the complete inverse images of $D \cap \mathcal{H}_A$ and $E \cap \mathcal{H}_A$. The image of $\widehat{D}_A \cup \widehat{E}_A$ under*

$$P_A(\sigma, m, n) = (\sigma, m)$$

is closed discrete. Every fibre is a finite union of arithmetic progressions in n, all parallel to the imaginary axis.

(b) *On* $\mathcal{H}_p = \mathcal{H} \cap \{p \neq 0\}$, *write* $p = e^{\rho}$. *Let* $\widehat{D}_p, \widehat{E}_p \subset \mathbb{C}^3_{x,y,\rho}$ *be the corresponding complete inverse images. The image of their union under*

$$P_p(\rho, m, n) = (\rho, m)$$

is closed discrete, and every fibre has the same finite-union-of- progressions description.

(c) *Let* $\Psi \in \mathrm{Aut}(\mathcal{H})$ *fix* m *and commute with* K*. On the logarithmic* A*-cover, the complete lift of* $\Psi(E) \cap \mathcal{H}_A$ *again has closed discrete* (σ, m)*-projection, with fibres finite unions of the same arithmetic progressions.*

Proof. We first treat D. Consider a compact subset of the (σ, m)-plane. For a lifted point of D, the quantities $A = e^{\sigma}$ and $M = e^{m}$ lie in compact annuli. Properness of M on the finite carrier union confines the downstairs base (h, a) to a compact set. Lemma 4.42, together with the bound on m, then confines the corresponding base point of B_K to a compact set. The function N_+ is bounded there, and hence $p = N_+/A$ is bounded. Since D_K is closed and discrete in the incidence space over B_K, only finitely many intermediate points occur. For each of them, only finitely many logarithms of A meet the chosen compact σ-set. The remaining base logarithms form one K-orbit and add no new (σ, m)-value. This proves that $P_A(\widehat{D}_A)$ is closed discrete.

The proof on the p-principal open is identical: bounded (ρ, m) puts $p = e^{\rho}$ and $M = e^{m}$ in compact annuli, the proper character and Lemma 4.42 confine the intermediate base, and $A = N_+/p$ is bounded. Closed discreteness of D_K then proves that $P_p(\widehat{D}_p)$ is closed discrete.

Fix either projected value. Properness and finiteness of M on every carrier leave only finitely many intermediate incidence points. The remaining base-logarithm changes satisfy $rk + sl = 0$ and hence have the form

$$(k, l) = \left(\frac{s}{d}t, -\frac{r}{d}t\right), \qquad t \in \mathbb{Z}.$$

They fix A and p in the character-adapted frame and change n by

$$-\frac{2\pi i(r^2 + s^2)}{d}\,t. \tag{4.98}$$

Thus each intermediate point contributes one arithmetic progression, and only finitely many occur.

We next treat the complete lift of one point of F_0. Choose one logarithm (x_0, y_0) of its base. Its possible m-values are

$$m_0 + 2\pi i(r\mathbb{Z} + s\mathbb{Z}) = m_0 + 2\pi i d\mathbb{Z}. \tag{4.99}$$

A compact subset of the m-plane meets only finitely many of these values. For fixed m, all remaining base logarithms differ by K and therefore represent one point of B_K. The adapted frame makes A, p constant along K, and only finitely many logarithms σ or ρ meet a compact set. This proves the closed-discrete projection statement for the lift of one finite point; the progression formula is (4.98). A finite union gives the assertion for E. The union with the corresponding projection of D is again closed discrete, since a compact set meets each of two closed discrete sets finitely.

Finally, if Ψ fixes m and commutes with K, it descends to the incidence space over B_K. For bounded m, only finitely many intermediate lifts of a point of F_0 occur by (4.99); Ψ sends each to one intermediate point. Passing to the logarithm of the transformed nonzero A-coordinate and then lifting the residual K-orbits gives exactly the same closed-discrete projection and progression conclusions. This proves part (c). □

Proposition 4.44. *The manifold*

$$\mathcal{H}_A \setminus (D \cup E)$$

is Oka.

Proof. Taking $A = e^{\sigma}$ and eliminating p gives an unramified covering

$$\mathbb{C}^3_{x,y,\sigma} \longrightarrow \mathcal{H}_A, \qquad p = f(x, y)e^{-\sigma}. \tag{4.100}$$

Its deletion set is the complete inverse image $\widehat{D}_A \cup \widehat{E}_A$. By Lemma 4.43(a), this set is closed discrete and its (σ, m)-projection is closed discrete. If it is infinite, the Rosay–Rudin discrete-projection criterion, Theorem 4.7, makes it tame; if it is finite, it is tame directly. The tame-subvariety complement theorem, Theorem 4.8, makes its complement in $\mathbb{C}^3$ elliptic, hence Oka. Because the deleted set is the complete inverse image, Theorem 4.1 proves the proposition. The wrong lines require no deletion here, since $L \subset \{A = 0\}$. □

Proposition 4.45. *The manifold*

$$\mathcal{H}_p \setminus (D \cup E \cup L)$$

is Oka.

Proof. Taking $p = e^{\rho}$ and eliminating A gives an unramified covering

$$\mathbb{C}^3_{x,y,\rho} \longrightarrow \mathcal{H}_p, \qquad A = f(x,y)e^{-\rho}. \tag{4.101}$$

Let $\widehat{D}_p, \widehat{E}_p$, and $\widetilde{L}$ denote the complete inverse images of D, E, L. The last of these is

$$\widetilde{L} = \bigcup_{k,l\in\mathbb{Z}} \{x = \alpha_w + 2\pi ik,\ y = \beta_w + 2\pi il,\ \rho \in \mathbb{C}\}. \tag{4.102}$$

It is a closed analytic union of lines. Since $D \cup E$ is ambiently closed discrete and disjoint from L, the union

$$\Sigma = \widehat{D}_p \cup \widehat{E}_p \cup \widetilde{L}$$

is a closed analytic subvariety of $\mathbb{C}^3$ of dimension at most one.

Let

$$B = P_p(\widehat{D}_p \cup \widehat{E}_p) \subset \mathbb{C}^2_{\rho,m}.$$

By Lemma 4.43(b), B is closed discrete and every n-fibre is a finite union of imaginary arithmetic progressions. The m-values on $\widetilde{L}$ form the closed discrete set

$$S_L = r\alpha_w + s\beta_w + 2\pi id\mathbb{Z}. \tag{4.103}$$

For fixed $m \in S_L$, the real parts of the corresponding n-values form a finite set.

For $m \in S_L$, the set

$$B^0_m = \{(\rho,m) \in B : |\rho| \le 2(|m|+1)\}$$

is finite. Choose $c_m > 0$ so large that

$$|\operatorname{Re} n + c_m| \ge 2(|m|+1) \tag{4.104}$$

for every real part occurring either on a wrong-line progression at level m or in a fibre over B^0_m. By Theorem 4.9, applied to the closed discrete set $S_L \subset \mathbb{C}$, there is an entire function φ with $\varphi(m) = c_m$ for $m \in S_L$. Theorem 4.10, with every prescribed multiplicity equal to one, gives an entire function g whose zero set is exactly S_L.

The subset

$$B' = \{(\rho,m) \in B : m \notin S_L\}$$

is closed discrete. For each $b = (\rho,m) \in B'$, choose $c_b > 0$ so large that

$$|\operatorname{Re} n + c_b| \ge 2(|m|+1) \tag{4.105}$$

for every n in the fibre over b. This is possible because only finitely many real parts occur in that fibre. Theorem 4.9, applied on $\mathbb{C}^2$ to B', gives an entire function G satisfying

$$G(\rho,m) = \frac{c_b - \varphi(m)}{g(m)} \quad \text{at } b = (\rho,m) \in B'.$$

Set

$$\psi(\rho,m) = \varphi(m) + g(m)G(\rho,m)$$

and obtain the triangular automorphism

$$\Theta(\rho,m,n) = (\rho, u = n + \psi(\rho,m), v = m). \tag{4.106}$$

We claim that

$$\Theta(\Sigma) \subset \Gamma = \{(\rho,u,v) \in \mathbb{C}^3 : |v| \le 1 + \|(\rho,u)\|\}. \tag{4.107}$$

Indeed, every point of Σ lies in exactly one of four classes:

(a) it belongs to $\widetilde{L}$;

(b) it belongs to $\widehat{D}_p \cup \widehat{E}_p$ and $m \notin S_L$;

(c) it belongs to $\widehat{D}_p \cup \widehat{E}_p$, one has $m \in S_L$, and $|\rho| \le 2(|m|+1)$;

(d) it belongs to $\widehat{D}_p \cup \widehat{E}_p$, one has $m \in S_L$, and $|\rho| > 2(|m|+1)$.

In the first and third cases, $\psi = c_m$ and (4.104) gives $|u| \geq 2(|m|+1)$. In the second case, (4.105) gives the same inequality. In the fourth case, $|m| < |\rho|/2$. Thus (4.107) holds in every case.

The projective closure of Γ misses the pure v-direction $[0:0:0:1]$ at infinity: a sequence tending to that point would satisfy $|v|/\|(\rho,u)\| \to \infty$, contrary to (4.107). Hence the projective closure of $\Theta(\Sigma)$ cannot contain the hyperplane at infinity. The analytic set Σ is therefore tame. Since

$$\dim \Sigma \leq 1 = 3-2,$$

Theorem 4.8 makes $\mathbb{C}^3 \setminus \Sigma$ elliptic and Oka. This complement is the complete inverse image of $\mathcal{H}_p \setminus (D \cup E \cup L)$ under (4.101). Theorem 4.1 proves the proposition. □

Let

$$\mathfrak{A} = \{A = p = 0\} \cap (\mathcal{H} \setminus L)$$

be the residual axis, and fix

$$e \in \mathfrak{A} \setminus (D \cup E).$$

Choose the pair (r,s) in Lemma 4.40 with all three additional properties at e, and use the corresponding character-adapted frame. All scalar coordinates and logarithmic projections in this subsection are understood in this newly chosen frame. The conclusions of Propositions 4.44 and 4.45 are intrinsic, and their proofs apply to every character satisfying the properness condition in Lemma 4.40; changing the frame here therefore does not change either principal-open conclusion.

Consider the vector field

$$Y = pW + (Wf)\partial_A, \qquad W = s\partial_x - r\partial_y. \tag{4.108}$$

Lemma 4.46. *The vector field Y is tangent to $\mathcal{H}$ and complete. The functions p and m are first integrals. Its flow is*

$$\begin{aligned} x_t &= x + tsp, \qquad\qquad y_t = y - trp, \qquad\qquad p_t = p,\\ A_t &= A + \frac{f(x+tsp,\, y-trp) - f(x,y)}{p}, \end{aligned} \tag{4.109}$$

where the quotient extends holomorphically to $p=0$ with value $t(Wf)(x,y)$. If $\alpha \in \mathcal{O}(\mathbb{C}^2_{p,m})$, then $\alpha(p,m)Y$ is complete and its time-one map is $\Phi_{\alpha(p,m)}$.

Proof. Tangency follows from

$$Y(Ap) = pWf = Y(f).$$

The base variables move in the constant direction pW, while p is fixed, which gives (4.109). The difference quotient is entire at $p=0$ by its power series. Hence the flow exists for all $t \in \mathbb{C}$. Finally, $Yp = Ym = 0$, so every $\alpha(p,m)$ is a holomorphic first integral. The replica assertion follows from Lemma 3.13. □

Split the universal lift into

$$D_+ = \{d \in D : |p(d)| \geq 1\}, \qquad D_- = \{d \in D : |p(d)| < 1\},$$

and set

$$\Pi_p = (p,m), \qquad \Pi_A = (A,m).$$

For the character chosen in this subsection, let Z_K be the corresponding character-kernel incidence cover and let $D_K \subset Z_K$ be the complete inverse image of D_0. Since the adapted frame is fixed by K, the functions A, p, m descend to Z_K. Write

$$\overline{D}_+ = D_K \cap \{|p| \geq 1\}, \qquad \overline{D}_- = D_K \cap \{|p| < 1\},$$

and denote the induced projections by

$$\overline{\Pi}_p = (p,m) : \overline{D}_+ \longrightarrow \mathbb{C}^2, \qquad \overline{\Pi}_A = (A,m) : \overline{D}_- \longrightarrow \mathbb{C}^2.$$

Lemma 4.47. *The maps $\overline{\Pi}_p$ and $\overline{\Pi}_A$ are proper. Consequently,*

$$B_+ = \Pi_p(D_+) = \overline{\Pi}_p(\overline{D}_+), \qquad B_- = \Pi_A(D_-) = \overline{\Pi}_A(\overline{D}_-)$$

are closed discrete subsets of $\mathbb{C}^2$. Every fibre of either universal projection Π_p or Π_A is a finite union of K-orbits. In the n-coordinate these orbits are arithmetic progressions with period direction (4.98). In particular, no properness is asserted for the universal projections on D_+ or D_-.

Proof. Let $Q \subset \mathbb{C}^2$ be compact. We first consider $\overline{\Pi}_p^{-1}(Q)$. On this set the coordinate p lies in a compact annulus because $|p| \geq 1$, while m remains bounded and hence $M = e^m$ lies in a compact annulus. Properness of M on the normalizations of the finitely many carriers confines the downstairs base (h, a) to a compact subset of their union. Lemma 4.42 then confines the corresponding base points in B_K to a compact set. The function N_+ is bounded there, and

$$A = \frac{N_+}{p}$$

is therefore bounded. Thus all points under consideration lie in a compact subset of the intermediate incidence space Z_K. Since D_K is closed discrete, only finitely many such points occur. Hence $\overline{\Pi}_p$ is proper.

Now consider $\overline{\Pi}_A^{-1}(Q)$. Boundedness of m, properness of M on the carriers, and Lemma 4.42 again confine the intermediate base to a compact set. On that set $|N_+|$ has a positive lower bound. Since $Ap = N_+$, boundedness of A and the condition $|p| < 1$ give a constant $\delta > 0$ such that $\delta \leq |p| < 1$; hence all such p lie in the compact annulus $\delta \leq |p| \leq 1$. All incidence coordinates consequently lie in a compact subset of Z_K. Closed discreteness of D_K again leaves only finitely many points, proving that $\overline{\Pi}_A$ is proper.

It follows that every compact subset of $\mathbb{C}^2$ meets either image B_+ or B_- in only finitely many points; hence both images are closed discrete. Finally, the covering

$$\mathcal{H} \longrightarrow Z_K$$

has deck group K, and $D_\pm$ are the complete inverse images of $\overline{D}_\pm$. Properness of the induced maps shows that a fixed projected value has only finitely many inverse images in $\overline{D}_\pm$. The full inverse image of each such intermediate point is one K-orbit. The adapted frame fixes A and p along that orbit, while (4.98) shows that n changes by

$$-\frac{2\pi i(r^2 + s^2)}{d}\, t, \qquad t \in \mathbb{Z}.$$

Thus every universal fibre is a finite union of the asserted arithmetic progressions. □

The set of m-values on L is the set S_L in (4.103). The separation clause in Lemma 4.40 gives

$$m_e = r x_e + s y_e \notin S_L. \tag{4.110}$$

Write

$$B_+^\circ = \{(p, m) \in B_+ : m \notin S_L\}.$$

It is closed discrete.

Lemma 4.48. *For every $b = (p, m) \in B_+^\circ$ and every $R > 0$, there is a time $t_b \in \mathbb{C}$ such that*

$$|A(\Phi_{t_b}(d))| > R \quad \textit{for every } d \in D_+ \textit{ with } \Pi_p(d) = b.$$

Proof. Write $T = pt$. By Lemma 4.47, a fibre has finitely many representatives modulo the arithmetic K-progressions. The latter have the same values of $h = e^x$, $a = e^y$, A, and p, so it is enough to treat those finitely many representatives. Formula (4.109) gives

$$pA_t = f(x + sT, y - rT) = ae^{-rT} - a_0 + 2h^3 e^{3sT} + 3\zeta^2 h e^{sT}. \tag{4.111}$$

The real exponents $-r, 0, 3s, s$ are pairwise distinct. Choose $\varepsilon \in \{1, -1\}$ such that the largest member of

$$\{\varepsilon(-r), 0, \varepsilon(3s), \varepsilon s\}$$

is positive. It is unique. Along $T = \varepsilon R'$ with $R' \to +\infty$, the corresponding term in (4.111) dominates. Its coefficient is one of $a, -a_0, 2h^3, 3\zeta^2 h$, and every one of these coefficients is nonzero. The same direction works for all the finitely many representatives, so their moduli tend uniformly to infinity. Taking R' sufficiently large and setting $t_b = \varepsilon R'/p$ proves the claim. □

Choose a proper exhaustion function $\chi : B_+^\circ \to [0, \infty)$ and, using Lemma 4.48, choose t_b such that

$$|A(\Phi_{t_b}(d))| > e^{\chi(b)} \quad (d \in D_+,\ \Pi_p(d) = b). \tag{4.112}$$

Let $g \in \mathcal{O}(\mathbb{C})$ have zero set exactly S_L. Since $B_+^\circ \subset \{|p| \geq 1\}$ and $m_e \notin S_L$, the set

$$B_+^\circ \cup \{(0, m_e)\}$$

is closed discrete. Theorem 4.9 gives $G \in \mathcal{O}(\mathbb{C}^2)$ satisfying

$$G(b) = \frac{t_b}{g(m_b)} \quad (b = (p_b, m_b) \in B_+^\circ), \qquad G(0, m_e) = \frac{c}{g(m_e)},$$

where $c \neq 0$ is fixed. The corresponding replica is

$$\alpha(p,m) = g(m)G(p,m), \qquad \phi = \Phi_{\alpha(p,m)}. \tag{4.113}$$

Proposition 4.49. *The map ϕ is an automorphism of $\mathcal{H}$ with the following properties:*

(i) *ϕ fixes L pointwise;*
(ii) *$\phi(e) \in \mathcal{H}_A$;*
(iii) *ϕ fixes m and commutes with K;*
(iv) *if $D' = \phi(D)$, then the complete logarithmic lift of $D' \cap \mathcal{H}_A$ has closed discrete (σ, m)-projection, and every fibre is a finite union of arithmetic progressions in the direction (4.98).*

Proof. The map is an automorphism by Lemma 4.46. Since $m(L) \subset S_L$, one has $\alpha = 0$ on L, proving (i). At e one has $p = 0$ and $\alpha(0, m_e) = c$. Formula (4.109) therefore gives

$$A(\phi(e)) = c(Wf)(e) \neq 0,$$

which proves (ii). Both the field and its time function preserve m. The K-action preserves h, a, A, p, m in the adapted frame, and hence preserves f, Y, and α. This proves (iii).

It remains to prove (iv). Suppose a sequence in the complete logarithmic lift of $D' \cap \mathcal{H}_A$ has (σ, m) in a compact set. Then m is bounded; moreover the transformed coordinate $A' = e^{\sigma}$ lies in a compact annulus. Pull the sequence back by ϕ to D.

For points in D_+ with $m \notin S_L$, the interpolation gives time t_b on the entire Π_p-fibre. The bound (4.112) and the upper bound for $|A'|$ show that $\chi(p, m)$ is bounded. Properness of χ leaves only finitely many values $b = (p, m) \in B_+^\circ$. By Lemma 4.47, the fibre over each such b has only finitely many representatives modulo K. Since ϕ commutes with K, the transformed value A' is constant on each residual K-orbit. A compact set in the σ-plane meets the logarithms of each fixed nonzero value of A' in only finitely many points. Consequently only finitely many projected (σ, m)-values occur.

For points in D_+ with $m \in S_L$, one has $\alpha = 0$ and $A' = A$. Boundedness of m confines the intermediate base to a compact carrier set. Since A lies in an annulus, $p = N_+/A$ is bounded; together with $|p| \geq 1$, this places (p, m) in a compact set. Closed discreteness of B_+ leaves only finitely many such projected values. The finite- representative and logarithm argument of the preceding paragraph again gives only finitely many projected (σ, m)-values.

Consider finally points of D_-. Boundedness of m confines the images in the intermediate cover to a compact carrier-base set. If $|p|$ is bounded below, then $A = N_+/p$ is bounded, so these images lie in a compact subset of the intermediate incidence space. Closed discreteness of D_K leaves only finitely many intermediate points and hence only finitely many projected logarithmic values.

Otherwise, after passing to a subsequence, $p \to 0$. Since $|p| < 1$ and m remains in a compact set, the entire function $\alpha(p, m)$ is bounded. With $t = \alpha(p, m)$, Taylor's formula applied to (4.109) gives, uniformly after choosing the intermediate representatives in the compact base set,

$$\frac{f(x + tsp, y - trp) - f(x, y)}{p} = O(1). \tag{4.114}$$

Thus $A' - A = O(1)$. Boundedness of A' would imply boundedness of A. The positive lower bound for $|N_+|$ and the identity $A = N_+/p$, however, give $|A| \to \infty$. This contradiction excludes $p \to 0$, reducing the small end to the preceding compactness argument.

For a fixed (σ, m), the same three cases leave only finitely many intermediate representatives. Since ϕ commutes with K, the full inverse image of each representative is one arithmetic progression in n with period direction (4.98). This proves the fibre assertion and completes the proof. □

Write

$$D' = \phi(D), \qquad E' = \phi(E).$$

Part (iii) of Proposition 4.49 and Lemma 4.43(c) show that the complete logarithmic lift of $E' \cap \mathcal{H}_A$ has closed discrete (σ, m)-projection and arithmetic fibres. Combining this with part (iv) of the proposition gives the same conclusion for $(D' \cup E') \cap \mathcal{H}_A$. Denote its complete inverse image under the logarithmic covering (4.100) by $\widehat{S}'$. This is a closed discrete subset of $\mathbb{C}^3$ whose (σ, m)-projection is closed discrete. If $\widehat{S}'$ is infinite, it is tame by Theorem 4.7; if it is finite, it is tame directly. Its dimension is zero, so Theorem 4.8

shows that $\mathbb{C}^3 \setminus \widehat{S}'$ is Oka. Since $\widehat{S}'$ is the complete inverse image of $(D' \cup E') \cap \mathcal{H}_A$, Theorem 4.1 gives

$$\mathcal{H}_A \setminus (D' \cup E') \quad \text{is Oka.} \tag{4.115}$$

The resulting neighbourhood is

$$\Omega_e = \phi^{-1}\big(\mathcal{H}_A \setminus (D' \cup E')\big). \tag{4.116}$$

This set is Oka and contains e. Indeed, $\phi(e) \in \mathcal{H}_A$, and injectivity of ϕ prevents $\phi(e)$ from lying in $D' \cup E'$. It avoids $D \cup E$ by definition. It also avoids L, since ϕ fixes L pointwise and $L \subset \{A = 0\}$. Finally, it is analytic-Zariski open in $\mathcal{H}$, because its complement is

$$\phi^{-1}\big(\{A = 0\} \cup D' \cup E'\big).$$

The set $D' \cup E'$ is closed discrete and therefore locally finite, so the displayed union is a closed analytic subvariety.

Proof of Theorem 4.39. Work first on

$$X_4 = \mathcal{H} \setminus (D \cup E \cup L).$$

The analytic-Zariski-open subsets

$$X_4 \cap \{A \neq 0\} = \mathcal{H}_A \setminus (D \cup E), \qquad X_4 \cap \{p \neq 0\} = \mathcal{H}_p \setminus (D \cup E \cup L)$$

are Oka by Propositions 4.44 and 4.45. They cover every point away from the residual axis. For each retained point on the axis, (4.116) gives an analytic-Zariski-open Oka neighbourhood contained in X_4. Hence every point of X_4 has an analytic-Zariski-open Oka neighbourhood, and Theorem 3.2 makes X_4 Oka.

Let $\pi : \mathcal{H} \to Z$ be the base change of the universal exponential covering $\epsilon : \mathbb{C}^2 \to T$. It is therefore an unramified surjective holomorphic covering. By construction,

$$D = \pi^{-1}(D_0), \qquad E = \pi^{-1}(F_0), \qquad L = \pi^{-1}(\ell).$$

Thus restriction gives the exact covering

$$\mathcal{H} \setminus (D \cup E \cup L) \longrightarrow U_4 \setminus (D_0 \cup F_0).$$

Theorem 4.1 descends the Oka property and proves the theorem. □

We finally record why this theorem covers the four exact order-four charts used later. The companion selected chart is

$$\langle B, p\rangle = a - q_-(h) = N_-, \qquad q_-(h) = a_0 + H(h).$$

Replacing H by $-H$, interchanging N_+ with N_-, and relabelling A as B leaves every step above unchanged. In particular, the exponential polynomial in (4.111) has the same four distinct exponents; only two nonzero coefficients change sign.

The sign involution on the global normalized double cover $\bar{Y}_{4,i}$ of Lemma 3.8 sends h to $-h$ and interchanges N_+ and N_-. On the common domain of the two displayed incidence coordinates, its projective-coordinate relation is

$$[\xi_- : \eta_-] = \left[\eta_+ : \frac{2ia}{\kappa_\zeta}\xi_+\right], \tag{4.117}$$

where

$$\kappa_\zeta = \frac{\Delta_4(z)}{z - \zeta}$$

and $2ia/\kappa_\zeta$ is a holomorphic unit on this common domain. The coefficient need not be a unit on either complete incidence model; (4.117) is only an overlap formula. The transport of complete charts instead follows from the global sign involution and the exact chart images established in Lemma 3.10: it maps each complete chart image to its sign-conjugate image, including the entire wrong-ruling line in the uncorrected model. As a biholomorphism of these complete images, it transports complete inverse images of deletion centres and preserves their ambient closedness. The companion carrier identity follows also by interchanging N_+ and N_- in (4.89). Thus Theorem 4.39 applies to all four corrected charts; the orbit-specific check is in Lemma 4.51.

4.5. **Bandwise and compact descent.** We now combine the regular-suspension theorem with the two corrected-chart theorems. The descent through the normalized quadratic cover uses the following lemma. It is formulated after a closed discrete deletion because that is the exact relative setting needed below.

Lemma 4.50. *Let a finite group G act holomorphically on a normal complex space $\bar{Z}$, and let $p : \bar{Z} \to Z \simeq \bar{Z}/G$ be the finite holomorphic quotient map to a complex manifold. Thus the set-theoretic fibres of p are precisely the G-orbits. Let $C, B \subset Z$ be disjoint closed analytic subsets, and assume that p is unramified over $Z \setminus B$. Let $S \subset Z$ be closed and discrete, and consider*

$$Z^\circ = Z \setminus S, \quad C^\circ = C \cap Z^\circ, \quad B^\circ = B \cap Z^\circ, \quad \bar{Z}^\circ = \bar{Z} \setminus p^{-1}(S).$$

Suppose that a G-invariant open submanifold

$$\bar{N}^\circ \subset p^{-1}(Z^\circ \setminus B^\circ)$$

contains $p^{-1}(C^\circ)$, has closed analytic complement in $\bar{Z}^\circ$, and is covered by analytic-Zariski-open Oka subsets of $\bar{N}^\circ$. Then C° has an analytic-Zariski-open Oka neighbourhood in Z°.

Proof. The restriction $p^\circ : \bar{Z}^\circ \to Z^\circ$ is again the finite quotient by G, with the same orbits as fibres. Localization applied to the stipulated analytic-Zariski-open cover makes $\bar{N}^\circ$ Oka. Lemma 3.19, applied to p° with $D = B^\circ$, $C = C^\circ$, and $\bar{N} = \bar{N}^\circ$, gives the claimed analytic-Zariski-open Oka neighbourhood of C° in Z°. □

Fix $m \in \{3, 4\}$. Let $D \subset U_m^{\mathrm{reg}}$ be a finite union of full residual-deck orbits through points of the common principal core $\mathcal{K}_m$, let $F \subset Y_m$ be finite, and set $S = D \cup F$. Removing from F the points which already lie in D does not change the complement, so we assume throughout that $D \cap F = \varnothing$.

Lemma 4.51. *For every exceptional curve C_i and every one of the four corrected incidence charts $\mathcal{C} \subset \bar{N}_i$, the deletion*

$$\mathcal{C} \cap \bar{p}_i^{-1}\big(S \cap Y_{m,i}\big)$$

is the disjoint union of a finite set and an ambient-closed discrete set contained in finitely many nonzero regular-carrier curves. Consequently,

$$\mathcal{C} \setminus \bar{p}_i^{-1}\big(S \cap Y_{m,i}\big) \quad \textit{is Oka.} \tag{4.118}$$

Proof. The inverse image of $F \cap Y_{m,i}$ is finite because $\bar{p}_i$ is finite. We treat the lift of D.

The complete inverse image of D under $\bar{p}_i$ is closed and discrete: the map is proper with finite fibres, and D is closed discrete in the whole band. Its intersection with a corrected chart is therefore closed and discrete relative to that chart. The local deletion theorem requires closedness in the uncorrected incidence model as well. The only additional boundary introduced by that model is the finite union of wrong-ruling lines, and the nonzero carrier identities exclude accumulation there, as we now check.

In order three, use $R = Ap$ in a selected plus chart. By (4.3), every lifted core orbit lies on one of finitely many carrier hypersurfaces

$$\mathscr{C}_\nu = \left\{R\big((4w^3 + 6w) - (1 + w^2)R\big) = c_\nu\right\}, \qquad c_\nu \neq 0.$$

If a sequence on such a carrier converged in the uncorrected model to a point of a wrong line, then $A \to 0$ while p stays finite, so $R = Ap \to 0$; the right-hand side would tend to zero, contradicting $c_\nu \neq 0$. Thus the lift is ambient closed and lies on finitely many nonzero carriers. The same calculation applies in the companion chart. The sign map (4.5) is a biholomorphism of the uncorrected product models, so it preserves both ambient closedness and the carrier property. Theorem 4.30 applies after adjoining the finite lift of F.

In order four, the carrier equation is

$$(a - a_0 + H(h))(a - a_0 - H(h)) = 2ia(z_\nu - \zeta), \qquad z_\nu^4 \neq 1,$$

and the selected incidence product is $\langle A, p\rangle = a - a_0 + H(h)$. Suppose that a lifted-orbit sequence converges in the uncorrected incidence model to the wrong line. There are only finitely many carrier values z_ν, so pass to a subsequence with fixed z_ν. At the limit $A = 0$ and p is finite; hence the first factor $\langle A, p\rangle$ tends to zero, while the second factor $a - a_0 - H(h)$ remains finite. The right-hand side tends to $2iq(h_w)(z_\nu - \zeta) \neq 0$, since $q(h_w) \neq 0$ at the wrong line and $z_\nu^4 \neq 1$. This contradiction proves ambient closedness. The companion and sign-transformed charts exchange the two factors as in Theorem 4.11(G6) and satisfy the same argument. Theorem 4.39 applies after adjoining the finite lift of F. This proves (4.118) in all four charts. □

Theorem 4.52. *Let $D \subset U_m^{\mathrm{reg}}$ be a finite union of full residual-deck orbits through common-principal-core points and let $F \subset Y_m$ be finite. Then:*

(i) *For every exceptional curve C_i, the manifold $Y_{m,i} \setminus ((D \cup F) \cap Y_{m,i})$ is Oka.*

(ii) *The manifold $Y_m \setminus (D \cup F)$ is Oka.*

Proof. Write $S = D \cup F$. By Theorem 4.28,

$$U_m^{\mathrm{reg}} \setminus S = U_m^{\mathrm{reg}} \setminus \big(D \cup (F \cap U_m^{\mathrm{reg}})\big)$$

is Oka.

Fix an exceptional curve C_i and work in $Y_{m,i} = U_m^{\mathrm{reg}} \cup C_i$. Set

$$S_i = S \cap Y_{m,i}, \qquad \bar{S}_i = \bar{p}_i^{-1}(S_i), \qquad \bar{Y}_{m,i}^\circ = \bar{Y}_{m,i} \setminus \bar{S}_i,$$

and consider

$$\bar{N}_i^\circ = \bar{N}_i \setminus \bar{S}_i.$$

Every corrected chart punctured by $\bar{S}_i$ is Oka by Lemma 4.51. These punctured charts are analytic-Zariski open relative to $\bar{N}_i^\circ$: if C is an unpunctured chart, then

$$\bar{N}_i^\circ \setminus (C \setminus \bar{S}_i) = (\bar{N}_i \setminus C) \cap \bar{Y}_{m,i}^\circ,$$

which is closed analytic in $\bar{N}_i^\circ$. Their union is precisely $\bar{N}_i^\circ$, because the unpunctured four-chart union is $\bar{N}_i$. Hence localization makes $\bar{N}_i^\circ$ Oka.

The set $\bar{N}_i^\circ$ is invariant under the sign group, contains

$$\bar{p}_i^{-1}(C_i \setminus S_i),$$

and lies over $(Y_{m,i} \setminus S_i) \setminus (D_{m,i} \setminus S_i)$. Its complement in $\bar{Y}_{m,i}^\circ$ is

$$(\bar{Y}_{m,i} \setminus \bar{N}_i) \cap \bar{Y}_{m,i}^\circ,$$

which is closed analytic. Apply Lemma 4.50 with

$$Z = Y_{m,i}, \quad C = C_i, \quad B = D_{m,i}, \quad S = S_i.$$

We obtain an analytic-Zariski-open Oka neighbourhood N_i° of $C_i \setminus S_i$ in $Y_{m,i} \setminus S_i$.

The two sets

$$U_m^{\mathrm{reg}} \setminus S_i \quad \text{and} \quad N_i^\circ$$

are analytic-Zariski-open Oka subsets of $Y_{m,i} \setminus S_i$ and cover it. Indeed, the first contains the complement of C_i, while the second contains the retained part of C_i. Localization gives $Y_{m,i} \setminus S_i$ Oka, proving (i).

Finally, the finitely many sets

$$Y_{m,i} \setminus S = (Y_{m,i} \setminus S_i)$$

are analytic-Zariski open in $Y_m \setminus S$ and cover it: their unpunctured complements are the unions of the exceptional curves other than C_i, and intersecting with $Y_m \setminus S$ preserves relative analyticity. A last application of localization proves (ii). ◻

The mixed band theorem now treats arbitrary finite subsets of the semistable manifolds. The key point is that a finite set downstairs has an infinite complete inverse image upstairs, but its intersection with one band has the exact mixed form of Theorem 4.52.

Lemma 4.53. *Let $Q \subset M_m$ be finite and set $\widetilde{Q} = \pi_m^{-1}(Q)$. For every translated band $Y = Y_m^{(r)}$, there is a disjoint decomposition*

$$\widetilde{Q} \cap Y = D_Y \sqcup F_Y, \tag{4.119}$$

where D_Y is a finite union of full residual-deck orbits through points of the common principal smooth core and is closed discrete in Y, while F_Y is a finite subset of the toroidal central-fibre part. Each toroidal orbit contributing to F_Y contributes at most two points.

Proof. The complete inverse image of one point of M_m under the regular cyclic covering is one Γ_m-orbit. Hence $\widetilde{Q}$ is a finite union of such orbits. The residual deck action preserves the common core and its toroidal complement. Every band contains the whole common core, so the intersection of a core orbit with Y is the full orbit. The set $\widetilde{Q}$ is closed discrete because it is the complete inverse image of a finite set under a covering; consequently its core part D_Y is closed discrete in the band. By Lemma 4.12, each toroidal orbit meets Y in at most two points. Since only finitely many orbits occur, their toroidal intersection F_Y is finite. The two invariant cases exhaust $\widetilde{Q} \cap Y$. □

Theorem 4.54. *For $m = 3, 4$ and every finite subset $Q \subset M_m$, the manifold $M_m \setminus Q$ is Oka.*

Proof. Let $\widetilde{Q} = \pi_m^{-1}(Q)$. For each translated band $Y_m^{(r)}$, Lemma 4.53 gives

$$\widetilde{Q} \cap Y_m^{(r)} = D_r \sqcup F_r$$

in the exact scope of Theorem 4.52. Therefore

$$Y_m^{(r)} \setminus \widetilde{Q} = Y_m^{(r)} \setminus (D_r \cup F_r)$$

is Oka.

These punctured bands cover $\widehat{M}_m \setminus \widetilde{Q}$. They are analytic-Zariski open relative to that punctured manifold, because

$$(\widehat{M}_m \setminus \widetilde{Q}) \setminus (Y_m^{(r)} \setminus \widetilde{Q}) = (\widehat{M}_m \setminus Y_m^{(r)}) \cap (\widehat{M}_m \setminus \widetilde{Q}),$$

and the first factor on the right is closed analytic in $\widehat{M}_m$. Localization gives $\widehat{M}_m \setminus \widetilde{Q}$ Oka.

Because $\widetilde{Q}$ is the complete inverse image of Q, the regular cyclic covering restricts to a surjective unramified holomorphic covering

$$\pi_m : \widehat{M}_m \setminus \widetilde{Q} \longrightarrow M_m \setminus Q.$$

Covering invariance proves that $M_m \setminus Q$ is Oka. □

Proof of the deletion assertion in Theorem 1.2. Let $P \subset X$ be finite. Its complete inverse images are

$$Q_4 = \varpi_4^{-1}(P \cap U_1) \subset M_4, \qquad Q_3 = \varpi_3^{-1}(P \cap U_2) \subset M_3.$$

They are finite because ϖ_4 and ϖ_3 are finite maps. Theorem 4.54 makes $M_4 \setminus Q_4$ and $M_3 \setminus Q_3$ Oka. Completeness of the inverse images gives the restricted surjective coverings

$$M_4 \setminus Q_4 \longrightarrow U_1 \setminus P, \qquad M_3 \setminus Q_3 \longrightarrow U_2 \setminus P.$$

They are unramified because the original finite maps are. Covering invariance therefore shows that both $U_1 \setminus P$ and $U_2 \setminus P$ are Oka.

These two sets cover $X \setminus P$. Moreover they are analytic-Zariski open in that punctured manifold, since

$$(X \setminus P) \setminus (U_j \setminus P) = S_j \cap (X \setminus P), \qquad j = 1, 2,$$

is closed analytic relative to $X \setminus P$. Kusakabe localization now gives that $X \setminus P$ is Oka. □

Throughout this module, every finite or closed discrete point centre is endowed with its reduced induced complex-subspace structure.

4.6. **Curves in a two-row band.** The exceptional curves $C_1, \ldots, C_m \subset Y_m$ are pairwise disjoint. The conic equation also defines the closed analytic curve $Z_m^0 = \{A = B = 0\} \subset \mathcal{X}_m$: it lies over the finitely many levels $\Delta_m(z) = 0$. We write

$$\mathfrak{A}_m = \rho_m^{-1}(Z_m^0)$$

for its complete inverse image, including the exceptional curves and the proper transforms of the remaining zero-axis components. The two following statements concern this fixed two-row band; no global curve in X_a is implicit in their formulation.

Lemma 4.55. *The two principal opens $Y_{m,A} = \rho_m^{-1}(\{A \neq 0\})$ and $Y_{m,B} = \rho_m^{-1}(\{B \neq 0\})$ are Oka.*

Proof. The resolution is an isomorphism away from the conifold origins, which lie in $\{A = B = 0\}$. A nonzero value of A uniquely determines $B = \Delta_m(z)/A$; hence $Y_{m,A} \simeq L_m^\times$. The base E_m is Oka. Indeed, its two Danielewski charts in Lemma 3.6 have equations $ab = f(c)$ with f having simple roots. The complete fields $a\partial_c + f'(c)\partial_b$, $b\partial_c + f'(c)\partial_a$, and $a\partial_a - b\partial_b$ span the tangent bundle of each chart. The two charts are algebraic-Zariski open and cover E_m, so Kusakabe localization applies. Since $L_m^\times \to E_m$ is a holomorphic principal $\mathbf{C}^*$-bundle, Oka fibre-bundle invariance [20] proves the claim for A. The argument for B is identical. □

Theorem 4.56. *Let $I \subset \{1, \dots, m\}$ and let D, F be as in Theorem 4.52. Then*

$$Z_I := Y_m \setminus \left(D \cup F \cup \bigcup_{i \in I} C_i \right)$$

is Oka.

Proof. Write $\Sigma = D \cup F$ and $\Sigma_j = \Sigma \cap Y_{m,j}$ for each j. The analytic-Zariski-open set

$$R := U_m^{\mathrm{reg}} \setminus \Sigma$$

is Oka by Theorem 4.28. For every $j \notin I$, the set

$$V_j := Y_{m,j} \setminus \Sigma_j$$

is Oka by Theorem 4.52(i). It is naturally an open subset of Z_I: the space $Y_{m,j}$ contains no exceptional curve other than C_j.

The complement of R in Z_I is the finite union

$$\bigcup_{j \notin I} (C_j \setminus \Sigma),$$

which is closed analytic relative to Z_I. Similarly,

$$Z_I \setminus V_j = \bigcup_{\substack{k \notin I \\ k \neq j}} (C_k \setminus \Sigma)$$

is closed analytic in Z_I. Thus R and the V_j, $j \notin I$, are analytic-Zariski-open Oka subsets. They cover Z_I, since every point of Y_m is either in U_m^{reg} or on exactly one exceptional curve. Apply Theorem 3.2. □

Taking $D = F = \varnothing$ in Theorem 4.56 shows that deleting any subcollection of the exceptional curves from Y_m still gives an Oka manifold.

Theorem 4.57. *The manifold*

$$Y_m \setminus \mathfrak{A}_m$$

is Oka.

Proof. By definition,

$$Y_m \setminus \mathfrak{A}_m = Y_{m,A} \cup Y_{m,B}.$$

Both principal opens are Oka by Lemma 4.55. They are analytic-Zariski open in the displayed complement, since their relative complements are the zero sets of the pulled-back holomorphic sections A and B. They cover because a point outside $\{A = B = 0\}$ has at least one nonzero coordinate. Apply localization. □

5. Point blow-ups and smooth curve centres

5.1. **Blow-up tools and complete centres.**

Definition 5.1. Let Z be a complex manifold and let $A \subset Z$ be a closed complex subspace with coherent ideal sheaf $\mathcal{I}_A$. The blow-up of Z along A is the relative analytic Proj

$$\mathrm{Bl}_A Z = \mathrm{Proj}_Z \left(\bigoplus_{k \geq 0} \mathcal{I}_A^k \right),$$

with its canonical proper holomorphic map $\pi : \mathrm{Bl}_A Z \to Z$. The inverse image $E = \pi^{-1}(A)$ is called the exceptional divisor. When A is a smooth submanifold of codimension $r \geq 2$, one has

$$E \simeq \mathbb{P}(N_{A/Z}), \qquad \mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}(N_{A/Z})}(-1).$$

When A is a locally finite reduced set of points, this construction agrees locally with the ordinary blow-up of a point.

The next three lemmas are included because every global descent in this paper uses them. In particular, the assertion about an unramified covering after blow-up is not treated as formal shorthand.

Lemma 5.2. *Let $A \subset Z$ be a closed complex subspace and let $V \subset Z$ be open. There is a canonical biholomorphism over V*

$$\pi^{-1}(V) \simeq \mathrm{Bl}_{A\cap V} V.$$

If V is analytic-Zariski open in Z, then $\pi^{-1}(V)$ is analytic-Zariski open in $\mathrm{Bl}_A Z$.

Proof. The ideal of $A \cap V$ in V is $\mathcal{I}_A|_V$. Therefore

$$\left(\bigoplus_{k\geq 0} \mathcal{I}_A^k\right)|_V = \bigoplus_{k\geq 0} (\mathcal{I}_A|_V)^k.$$

Relative Proj commutes with restriction to an open subset of the base, which gives the asserted biholomorphism. If $Z \setminus V$ is closed analytic, then

$$(\mathrm{Bl}_A Z) \setminus \pi^{-1}(V) = \pi^{-1}(Z \setminus V)$$

is closed analytic, since the inverse image of a closed analytic subspace by a holomorphic map is closed analytic. □

Lemma 5.3. *Let $g : Z' \to Z$ be a flat holomorphic map, let $A \subset Z$ be a closed complex subspace, and let $A' = Z' \times_Z A$ be its scheme-theoretic inverse image. Then there is a canonical Cartesian square*

$$\begin{tikzcd} \mathrm{Bl}_{A'} Z' \arrow[r] \arrow[d] & \mathrm{Bl}_A Z \arrow[d] \\ Z' \arrow[r, "g"] & Z. \end{tikzcd}$$

Proof. Flatness gives $g^*(\mathcal{I}_A^k) = (g^*\mathcal{I}_A)^k = \mathcal{I}_{A'}^k$ for every $k \geq 0$. Thus the pullback of the Rees algebra of $\mathcal{I}_A$ is the Rees algebra of $\mathcal{I}_{A'}$. Relative Proj commutes with flat base change, which gives the Cartesian square. □

Lemma 5.4. *Let $g : Z' \to Z$ be a surjective unramified holomorphic covering and let $A \subset Z$ be a closed complex subspace. Give $A' = g^{-1}(A)$ the pulled-back complex-space structure. Then the map supplied by Lemma 5.3,*

$$\widetilde{g} : \mathrm{Bl}_{A'} Z' \longrightarrow \mathrm{Bl}_A Z,$$

is a surjective unramified holomorphic covering. If g has finitely many sheets, so does $\widetilde{g}$, with the same degree.

Proof. An unramified holomorphic covering is locally a disjoint union of biholomorphisms and is therefore flat. The Cartesian square follows from Lemma 5.3. Fix a point $z \in Z$ and an evenly covered neighbourhood U of z. Write

$$g^{-1}(U) = \coprod_{\alpha\in I} U_\alpha, \qquad g|_{U_\alpha} : U_\alpha \xrightarrow{\sim} U.$$

By Lemma 5.2, the inverse image of $\pi^{-1}(U) \subset \mathrm{Bl}_A Z$ is the disjoint union of the spaces

$$\mathrm{Bl}_{A'\cap U_\alpha} U_\alpha,$$

and every one of them is biholomorphic to $\mathrm{Bl}_{A\cap U} U$. Hence $\pi^{-1}(U)$ is evenly covered. Surjectivity and the statement about the degree follow immediately. □

Lemma 5.5. *Let $\Phi : Z \to Z'$ be a biholomorphism and let $A \subset Z$ be a closed complex subspace. Then Φ induces a unique biholomorphism*

$$\mathrm{Bl}_A Z \xrightarrow{\sim} \mathrm{Bl}_{\Phi(A)} Z'$$

which commutes with the two blow-down maps.

Proof. The pullback by Φ^{-1} identifies the ideal sheaf of A with the ideal sheaf of $\Phi(A)$ and therefore identifies their Rees algebras. Taking relative Proj gives the asserted map. Applying the same construction to Φ^{-1} gives its inverse. □

We state the precise external result needed for point centres. The word "tame" has the meaning fixed in the definition preceding Theorem 4.7.

Theorem 5.6. *Let $n > 1$ and let $D \subset \mathbf{C}^n$ be a closed discrete set.*

(i) *If D is infinite and tame, meaning that an automorphism of $\mathbf{C}^n$ carries it onto $\mathbb{N} \times \{0\} \subset \mathbf{C} \times \mathbf{C}^{n-1}$, then $\mathrm{Bl}_D \mathbf{C}^n$ is weakly subelliptic and hence Oka.*

(ii) *If D is finite, then $\mathrm{Bl}_D \mathbf{C}^n$ is of Class $\mathscr{A}$ and hence Oka.*

The infinite-centre statement is the tame discrete blow-up theorem [20, Proposition 6.4.12]; its Oka conclusion is also Kusakabe's [21, Proposition 4.10]. The finite-centre statement follows from Gromov's preservation of Class $\mathscr{A}$ under finitely many point blow-ups, as recorded in [27, p. 152]. Tameness matters in the infinite case: Kusakabe also gives closed discrete centres with non-Oka blow-ups.

Corollary 5.7. *Let $q : \mathbf{C}^n \to V$ be a surjective unramified holomorphic covering, let $S \subset V$ be closed and discrete, and suppose that the complete inverse image $q^{-1}(S)$ is tame. Then $\mathrm{Bl}_S V$ is Oka.*

Proof. Theorem 5.6 makes $\mathrm{Bl}_{q^{-1}(S)} \mathbf{C}^n$ Oka. By Lemma 5.4, this manifold covers $\mathrm{Bl}_S V$ unramifiedly. Covering invariance, Theorem 4.1, descends the Oka property. □

Corollary 5.8. *In Corollary 5.7, assume $n = 3$. It is enough to find a surjective complex-linear projection $L : \mathbf{C}^3 \to \mathbf{C}^2$ such that $L(q^{-1}(S))$ is closed and discrete.*

Proof. If $q^{-1}(S)$ is infinite, the Rosay–Rudin criterion, Theorem 4.7, makes it tame. A finite set is tame by definition. Apply Corollary 5.7. □

Lemma 5.9. *Let Z be a complex manifold and let $S \subset Z$ be closed and discrete. Assume:*

(i) *$Z \setminus S$ is Oka;*

(ii) *for every $s \in S$ there is an analytic-Zariski-open neighbourhood $V_s \subset Z$ of s such that $\mathrm{Bl}_{S\cap V_s} V_s$ is Oka.*

Then $\mathrm{Bl}_S Z$ is Oka.

Proof. Let $\pi : \mathrm{Bl}_S Z \to Z$ be the blow-down and set $E = \pi^{-1}(S)$. By Lemma 5.2,

$$(\mathrm{Bl}_S Z) \setminus E \simeq Z \setminus S$$

is analytic-Zariski open and Oka. For $s \in S$, the inverse image $\pi^{-1}(V_s)$ is canonically $\mathrm{Bl}_{S\cap V_s} V_s$; it is analytic-Zariski open, is Oka, and contains the complete exceptional fibre $\pi^{-1}(s)$. These opens cover $\mathrm{Bl}_S Z$. Kusakabe localization, Theorem 3.2, proves the claim. □

The same statement remains true if V_s is obtained by pulling back a principal open through a biholomorphism of Z: use Lemma 5.5 before applying Lemma 5.9.

5.2. **Scalar and corrected incidence charts.** We retain the notation of Subsection 4.3. Thus $T = (\mathbf{C}^*)^2$, $g \in \mathbf{C}[u^{\pm 1}, v^{\pm 1}]$, and

$$S_g = \{(q, A, B) \in T \times \mathbf{C}^2 : AB = g(q)\}^{\mathrm{reg}}.$$

Let $D \subset S_g$ be closed and discrete, with base image contained in a finite union of algebraic curves $C_\nu \subset T$ on which $g|_{C_\nu} = c_\nu \in \mathbf{C}^*$, and let $F \subset S_g$ be finite. Set

$$S = D \cup F.$$

Theorem 4.21 says that $S_g \setminus S$ is Oka. We now treat the points above S in the blow-up.

Lemma 5.10. *The two manifolds*

$$\mathrm{Bl}_{S\cap S_A} S_A, \qquad \mathrm{Bl}_{S\cap S_B} S_B,$$

where $S_A = S_g \cap \{A \neq 0\}$ and $S_B = S_g \cap \{B \neq 0\}$, are Oka.

Proof. We prove the assertion for S_A; the other follows by interchanging A and B. The covering

$$\pi_A : \mathbf{C}^3_{x,y,\sigma} \longrightarrow S_A, \quad (u, v, A, B) = (e^x, e^y, e^\sigma, g(e^x, e^y)e^{-\sigma})$$

is the universal logarithmic covering of $S_A \simeq (\mathbf{C}^*)^3$. Let $\widehat{S}_A$ be the complete inverse image of $S \cap S_A$. In the proof of Lemma 4.22, a proper character $M = u^r v^s$ was chosen and it was shown, with all logarithmic sheets retained, that the projection

$$(x, y, \sigma) \longmapsto (m, \sigma) = (rx + sy, \sigma)$$

maps $\widehat{S}_A$ to a closed discrete subset of $\mathbf{C}^2$. We recall the compactness argument because it is exactly the point needed for the blow-up.

Suppose (m, σ) remains in a compact subset of $\mathbf{C}^2$. Then $M = e^m$ and $A = e^\sigma$ lie in compact annuli. Properness of M on the normalizations of the finitely many carrier curves confines the base point to a compact part of those curves. On C_ν the equation is $AB = c_\nu$, so $B = c_\nu/A$ also lies in a compact annulus. The points of the closed discrete set D under consideration therefore lie in a fixed compact subset of S_A and are finite in number. For each such point the possible projected logarithms form a product of two arithmetic progressions, which meets a compact set in finitely many points. The lift of the finite set F is a finite union of the same type of product lattices. Hence the complete projected image is closed and discrete.

Corollary 5.8 now applies to $\pi_A : \mathbf{C}^3 \to S_A$ and proves that $\mathrm{Bl}_{S\cap S_A} S_A$ is Oka. □

The residual axis of S_g is the set of smooth points with $A = B = 0$. Carrier points never lie there because $AB = c_\nu \neq 0$, so an element of the centre on the axis necessarily belongs to the finite set F.

Lemma 5.11. *Let $s = (q_s, 0, 0) \in F$ be a smooth residual-axis point. There is an automorphism $\Phi \in \mathrm{Aut}(S_g)$ with the following properties:*

(a) *$A(\Phi(s)) \neq 0$;*
(b) *for $S' = \Phi(S)$, the complete inverse image of $S' \cap S_A$ in the logarithmic covering of S_A has closed discrete (m, σ)-projection.*

Proof. Choose the axis-adapted proper character from Lemma 4.15. With

$$W = su\frac{\partial}{\partial u} - rv\frac{\partial}{\partial v}, \qquad M = u^r v^s,$$

the complete suspension field

$$Y = BW + (Wg)\frac{\partial}{\partial A}$$

has first integrals B and M. Its flow Φ_t is given in Lemma 4.23. Since $(Wg)(q_s) \neq 0$, a nonzero time moves s into $\{A \neq 0\}$.

For the infinite carrier part, split D into $|B| \geq 1$ and $|B| < 1$ as in the proof of Proposition 4.26. On the first part the projection (B, M) has closed discrete image and finite fibres. Along every such fibre, the expression

$$BA_t = g(\Psi_{tB}(q))$$

is an exponential polynomial with pairwise distinct weights. Choosing a suitable real ray in the time variable makes one term dominate uniformly on the finite fibre. Prescribe times on the resulting closed discrete set so that $|A_t|$ exceeds a proper exhaustion threshold. Adjoin the single projected value $(0, M(q_s))$ and prescribe there a nonzero time. Cartan interpolation, Theorem 4.9, gives an entire first integral $\alpha(B, M)$ with all these values. The replica

$$\Phi = \Phi_{\alpha(B,M)}$$

is an automorphism.

The proof of Proposition 4.26 now applies verbatim, but we record its two estimates. On the large end, a bounded transformed logarithm σ' bounds the proper exhaustion and therefore leaves only finitely many projected fibres. On the small end, bounded m confines the base to a compact carrier set and boundedness of α there gives

$$A' - A = \alpha(B, M) \int_0^1 (Wg)(\Psi_{\tau\alpha(B,M)B}(q))\, d\tau = O(1).$$

If $B \to 0$, then $|A| = |c_\nu / B| \to \infty$, contradicting boundedness of A'. Thus B is bounded away from zero, and closed discreteness leaves only finitely many points. The finite set F contributes only finitely many product lattices. This proves (b), while the prescribed nonzero time proves (a). □

Theorem 5.12. *Under the hypotheses above, the manifold*

$$\mathrm{Bl}_S\, S_g$$

is Oka.

Proof. The complement of the exceptional divisor is $S_g \setminus S$, which is Oka by Theorem 4.21. Let $s \in S$. If $A(s) \neq 0$, Lemma 5.10 gives an Oka analytic-Zariski-open neighbourhood of the exceptional fibre over s. The case $B(s) \neq 0$ is symmetric.

If $A(s) = B(s) = 0$, use Lemma 5.11. Write $S' = \Phi(S)$. By the projected-centre criterion, $\mathrm{Bl}_{S' \cap S_A} S_A$ is Oka. By Lemma 5.5, its pullback through the induced biholomorphism of blow-ups is an Oka analytic-Zariski-open neighbourhood of the exceptional fibre over s. Lemma 5.9 completes the proof. □

We next adapt the order-three and order-four corrected chart arguments. The common logarithmic model is

$$\mathcal{H} = \{(x, y, A, p) \in \mathbf{C}^4 : Ap = f(x, y)\},$$

with a locally finite union L of wrong-ruling affine lines removed. The carrier part of the centre has $A \neq 0$ and $p \neq 0$; only the finite part can meet the residual axis $\{A = p = 0\}$.

Lemma 5.13. *Let $T \subset \mathcal{H} \setminus L$ be closed and discrete. Assume that, after writing $A = e^{\sigma}$ on*

$$V_A = (\mathcal{H} \setminus L) \cap \{A \neq 0\},$$

the complete inverse image of $T \cap V_A$ in $\mathbf{C}^3_{x,y,\sigma}$ has closed discrete image under the surjective linear projection $(x, y, \sigma) \mapsto (\sigma, m)$. Then

$$\mathrm{Bl}_{T \cap V_A} V_A$$

is Oka.

Proof. Eliminating p gives an unramified covering $\mathbf{C}^3_{x,y,\sigma} \to V_A$; here $L \subset \{A = 0\}$ in both corrected models. The stated projection hypothesis and Corollary 5.8 prove the assertion. Notice that the hypothesis concerns the complete inverse image, so no kernel-lattice branch is discarded. ◻

Lemma 5.14. *Let $\widetilde{S} \subset \mathcal{H} \setminus L$ be the complete logarithmic lift of the mixed centre in the corresponding corrected-chart theorem, and let $e \in \widetilde{S}$ satisfy $A(e) = 0$. In either the order-three or order-four model there is an automorphism $\Phi \in \mathrm{Aut}(\mathcal{H} \setminus L)$ such that*

(a) *$A(\Phi(e)) \neq 0$;*

(b) *Φ fixes every component of L pointwise;*

(c) *the complete further logarithmic lift of $\Phi(\widetilde{S}) \cap \{A \neq 0\}$ has closed discrete (σ, m)-projection.*

Proof. The point e belongs to the finite part of the centre, since the nonzero carrier identity excludes $A = 0$. Its base point lies on the smooth zero level $f = 0$ and, because $e \notin L$, differs from every wrong-line base point. The character-selection arguments used for an axis point depend only on this base point and not on the value of $p(e)$. We may therefore choose the character so that $(Wf)(e) \neq 0$ and so that $m(e)$ is not a wrong-line level.

In order three use the complete field

$$Y = p(b\partial_x - \partial_y) - e^y \partial_A, \qquad m = x + by.$$

In order four use

$$Y = p(s\partial_x - r\partial_y) + (Wf)\partial_A, \qquad m = rx + sy.$$

The character-selection lemmas ensure that the m-value of e does not belong to the closed discrete set S_L of wrong-line levels and, in order four, that $(Wf)(e) \neq 0$. Let γ be an entire function whose zero set is exactly S_L. The replica coefficient is chosen in the form

$$\alpha(p, m) = \gamma(m) G(p, m).$$

It therefore vanishes on L and its flow fixes L pointwise.

On the large carrier end, let B_+ be the closed discrete image of the projection (p, m) and consider

$$B_+^\circ = \{(p, m) \in B_+ : m \notin S_L\}.$$

Only on B_+° do we prescribe the escape times from (4.69) or (4.112); on the complementary fibres the factor $\gamma(m)$ forces the time to be zero. We verify explicitly that the time imposed at e can be chosen to move it. In order three, where $f = e^y + c$ and $e^y = -c$, the flow gives

$$A(\Phi_t(e)) = \begin{cases} \dfrac{c(1 - e^{-p(e)t})}{p(e)}, & p(e) \neq 0, \\ ct, & p(e) = 0. \end{cases}$$

In order four it gives

$$A(\Phi_t(e)) = \begin{cases} \dfrac{f(x_e + sp(e)t, y_e - rp(e)t)}{p(e)}, & p(e) \neq 0, \\ t(Wf)(e), & p(e) = 0. \end{cases}$$

The first order-four function is not identically zero because its derivative at $t = 0$ is $(Wf)(e) \neq 0$. Thus in both orders the bad times for which the displayed value vanishes form a discrete subset of $\mathbb{C}$.

Since $m(e) \notin S_L$, we may adjoin $(p(e), m(e))$ to B_+° and prescribe a time outside this bad set. If that projected value already lies in B_+°, the exponential-polynomial estimate used in the simultaneous-escape lemma holds on a sufficiently far tail of the chosen real ray. Removing the discrete bad times still leaves a time on that tail, so the same value both moves e and sends the complete carrier fibre beyond its escape threshold. Cartan interpolation on the resulting closed discrete set, with prescribed values divided by the nonzero factors $\gamma(m)$, determines G. The resulting wrong-line-fixing replica moves e into the A-principal open.

We spell out the missing case in the local-finiteness argument. On the large end with $m \notin S_L$, the escape threshold leaves only finitely many points of B_+°. On the large end with $m \in S_L$, the replica is the identity. If (σ, m) remains compact, then $A = e^\sigma$ lies in a compact annulus and properness of the character confines the carrier base to a compact set. The nonzero carrier factor is bounded there, so $p = f/A$ is bounded; together with $|p| \geq 1$, this confines the corresponding intermediate points to a compact subset. Closed discreteness leaves only finitely many of them. On the small end the Taylor difference quotient gives $A' - A = O(1)$, whereas the nonzero carrier factor gives $|A| \to \infty$ if $p \to 0$; hence p is bounded away from zero and the same compactness argument applies. These are exactly the three cases in Proposition 4.36 and Proposition 4.49.

The finite part produces only finitely many intermediate lifts for bounded m, and the remaining kernel branches change only the unused coordinate n. In order four this assertion after the replica is precisely Lemma 4.43(c). Thus the complete further lift has closed discrete (σ, m)-projection. □

Theorem 5.15. *Let Z be one of the four corrected order-three charts or one of the four corrected order-four charts appearing in Theorem 4.11. Let $S = D \cup F \subset Z$, where D is the ambient-closed regular-carrier set and F is finite, as in the corresponding mixed-deletion theorem. Then*

$$\mathrm{Bl}_S Z$$

is Oka.

Proof. The order-three and order-four charts require different covering arguments. We treat them separately.

First consider the order-three charts. After transporting by the exact companion or sign biholomorphism, write the chart in the standard form

$$Z_3 = (\mathbf{C}_w^* \times \mathbf{C}_A \times \mathbf{C}_p) \setminus \bigcup_j \ell_j, \qquad R = Ap.$$

The set S is countable. Choose

$$c \in \mathbf{C}^* \setminus R(S).$$

Then the analytic-Zariski-open set

$$V_c = Z_3 \cap \{R \neq c\}$$

contains the complete centre S. With $u = R - c$, it is exactly the shifted corrected chart $Z_{3,c}$ from (4.46). Its universal base-logarithmic covering is

$$q_c : \mathcal{H}_c \setminus L \longrightarrow V_c, \qquad \mathcal{H}_c = \{(x, y, A, p) : Ap = e^y + c\},$$

where L is the complete locally finite inverse image of the wrong-ruling lines. Let $\widetilde{S} = q_c^{-1}(S)$.

We first prove that $\mathrm{Bl}_{\widetilde{S}}(\mathcal{H}_c \setminus L)$ is Oka. Its complement of the exceptional divisor is $(\mathcal{H}_c \setminus L) \setminus \widetilde{S}$, an unramified covering of $V_c \setminus S$. The latter is Oka by Theorem 4.37, so covering invariance makes the former Oka. Let $e \in \widetilde{S}$. If $A(e) \neq 0$, the exceptional fibre over e has an analytic-Zariski-open Oka neighbourhood by Lemma 5.13; its projection hypothesis is the closed-discreteness statement proved in Proposition 4.32. If $A(e) = 0$, then e belongs to the finite part of the centre, because the nonzero carrier identity excludes $A = 0$. Lemma 5.14 gives an automorphism of $\mathcal{H}_c \setminus L$ which fixes L pointwise, moves e into $\{A \neq 0\}$, and supplies exactly the projection hypothesis of Lemma 5.13 for the transformed centre. Functoriality of blow-up under biholomorphisms and Lemma 5.13 therefore give an Oka analytic-Zariski-open neighbourhood of the complete exceptional fibre over e. Lemma 5.9 proves that the lifted blow-up is Oka.

Since $\widetilde{S}$ is the complete inverse image of S, Lemma 5.4 gives an unramified covering

$$\mathrm{Bl}_{\widetilde{S}}(\mathcal{H}_c \setminus L) \longrightarrow \mathrm{Bl}_S V_c.$$

Hence $\mathrm{Bl}_S V_c$ is Oka. It is analytic-Zariski open in $\mathrm{Bl}_S Z$, contains the entire exceptional divisor, and has complement equal to the inverse image of the analytic hypersurface $\{R = c\}$. The other open $(\mathrm{Bl}_S Z) \setminus \mathrm{Exc}(\mathrm{Bl}_S Z/Z) \simeq Z \setminus S$ is Oka by the order-three mixed-deletion theorem. These two opens cover $\mathrm{Bl}_S Z$; Kusakabe localization proves the order-three assertion.

Now consider the order-four charts. The corrected intrinsic incidence chart is globally covered by the universal exponential pullback

$$q : \mathcal{H} \setminus L \longrightarrow Z, \qquad \mathcal{H} = \{(x, y, A, p) : Ap = e^y - q(e^x)\}.$$

Let $\widetilde{S} = q^{-1}(S)$. The complement $(\mathcal{H} \setminus L) \setminus \widetilde{S}$ is Oka because it covers the order-four mixed-deletion complement. For a centre point with $A \neq 0$, use Lemma 5.13; its projection hypothesis follows from Lemma 4.43(a). For a centre point with $A = 0$, use Lemma 5.14; the nonzero carrier identity again shows that such a point belongs to the finite part, and clause (c) of that lemma gives the projection hypothesis for the transformed centre. Exceptional-fibre localization makes $\mathrm{Bl}_{\widetilde{S}}(\mathcal{H} \setminus L)$ Oka. Blow-up base change gives an unramified covering of $\mathrm{Bl}_S Z$, and covering invariance completes the order-four case. □

5.3. **Bandwise and compact blow-ups.**

Lemma 5.16. *Let a finite group G act holomorphically on a normal complex space $\bar{Z}$, and let*

$$p : \bar{Z} \longrightarrow Z \simeq \bar{Z}/G$$

be the finite quotient map to a complex manifold. Let $B, C \subset Z$ be disjoint closed analytic subsets and assume that p is unramified over $Z \setminus B$. Let $S \subset Z$ be closed and discrete. Suppose that a G-invariant open submanifold

$$\bar{N} \subset p^{-1}(Z \setminus B)$$

contains $p^{-1}(C)$ and has closed analytic complement in $\bar{Z}$. Write

$$A = \bar{Z} \setminus \bar{N}, \qquad N = Z \setminus p(A).$$

Then N is analytic-Zariski open, $C \subset N \subset Z \setminus B$, and $p^{-1}(N) = \bar{N}$. If

$$\mathrm{Bl}_{p^{-1}(S \cap N)} \bar{N}$$

is Oka, then $\mathrm{Bl}_{S \cap N} N$ is Oka. Under the canonical open embedding $\mathrm{Bl}_{S \cap N} N \subset \mathrm{Bl}_S Z$, it is an analytic-Zariski-open neighbourhood of every exceptional fibre lying over $S \cap C$.

Proof. Lemma 3.19, with $D = B$, gives all the assertions about N and the finite unramified covering $p : \bar{N} \to N$. The centre $p^{-1}(S \cap N)$ is the complete inverse image of $S \cap N$. By unramified base change for blow-ups, Lemma 5.4, the induced map

$$\mathrm{Bl}_{p^{-1}(S \cap N)} \bar{N} \longrightarrow \mathrm{Bl}_{S \cap N} N$$

is an unramified covering. Covering invariance descends the Oka property. Finally, blow-up commutes with restriction to the open set N, so $\mathrm{Bl}_{S \cap N} N$ is the inverse image of N in $\mathrm{Bl}_S Z$. Its complement is the inverse image of the closed analytic set $Z \setminus N$, hence it is analytic-Zariski open. Since $C \subset N$, it contains the full exceptional fibre over every point of $S \cap C$. □

Theorem 5.17. *Let $D \subset U_m^{\mathrm{reg}}$ be a finite union of full residual-deck orbits through common-principal-core points, and let $F \subset Y_m$ be finite. For $S = D \cup F$, the blow-up*

$$\mathrm{Bl}_S Y_m$$

is Oka for $m = 3, 4$.

Proof. Write

$$U = U_m^{\mathrm{reg}}, \qquad S_U = S \cap U = D \cup (F \cap U).$$

We first prove that

$$\mathrm{Bl}_{S_U} U \quad \text{is Oka.} \tag{5.1}$$

The complement $U \setminus S_U$ is Oka by Theorem 4.28. Fix $s \in S_U$, and let $q \in E_m$ be its base point. Lemma 4.27 gives an algebraic Zariski-open neighbourhood

$$Q \simeq \mathbf{C}_u^* \times \mathbf{C}_c$$

of q on which $\mathcal{L}_m$ is holomorphically trivial. Since S_U is countable, there is a value

$$c_0 \notin c\big(\mathrm{pr}(S_U) \cap Q\big), \qquad c_0 \neq c(q).$$

Set

$$T = Q \setminus \{c = c_0\} \simeq \mathbf{C}_u^* \times \mathbf{C}_v^*, \qquad v = c - c_0,$$

and let $V_s = U|_T$. This is an analytic-Zariski-open neighbourhood of s in U. In a holomorphic frame of $\mathcal{L}_m|_Q$, it is the regular scalar suspension

$$V_s = \{AB = g(u, v)\}^{\mathrm{reg}}, \qquad g = \Delta_m(z)|_T \in \mathbf{C}[u^{\pm 1}, v^{\pm 1}].$$

Indeed, in the product coordinates supplied by Lemma 4.27, the regular function z belongs to $\mathbf{C}[u, u^{-1}, c]$; substituting $c = v + c_0$ and applying the polynomial Δ_m gives the displayed Laurent polynomial.

The set $D \cap V_s$ is closed and discrete, and its base image is contained in the finitely many reduced curves $(C_\nu \cap T)_{\text{red}}$, on each of which $g = \Delta_m(z_\nu) = c_\nu \neq 0$. The set $F \cap V_s$ is finite. Thus all hypotheses of Theorem 5.12 hold, and $\mathrm{Bl}_{S_U \cap V_s} V_s$ is Oka. Applying Lemma 5.9 to U and S_U proves (5.1).

Fix an exceptional curve C_i and set

$$Y_{m,i} = U \cup C_i, \qquad S_i = S \cap Y_{m,i}.$$

It remains to obtain a neighbourhood of the exceptional fibres over $S_i \cap C_i = F \cap C_i$. The normalized Galois double cover $\bar{p}_i : \bar{Y}_{m,i} \to Y_{m,i}$ is unramified over the complement of $D_{m,i}$, and $D_{m,i} \cap C_i = \varnothing$. The four corrected incidence charts form a sign-invariant open submanifold $\bar{N}_i$ which contains $\bar{p}_i^{-1}(C_i)$, lies in $\bar{p}_i^{-1}(Y_{m,i} \setminus D_{m,i})$, and has closed analytic complement. The descended open set is

$$N_i = Y_{m,i} \setminus \bar{p}_i(\bar{Y}_{m,i} \setminus \bar{N}_i).$$

Lemma 5.16 gives $C_i \subset N_i$, $\bar{p}_i^{-1}(N_i) = \bar{N}_i$, and shows that N_i is analytic-Zariski open in $Y_{m,i}$.

Set

$$S_{i,N} = S_i \cap N_i, \qquad \bar{S}_{i,N} = \bar{p}_i^{-1}(S_{i,N}).$$

By Lemma 4.51, the centre induced in each of the four corrected charts has the ambient-closed regular-carrier part and finite part required by Theorem 5.15. Thus every chart blow-up is Oka. These four blow-ups are analytic-Zariski open in $\mathrm{Bl}_{\bar{S}_{i,N}} \bar{N}_i$ and cover it, so localization gives

$$\mathrm{Bl}_{\bar{S}_{i,N}} \bar{N}_i \quad \text{Oka.}$$

Applying Lemma 5.16 with $Z = Y_{m,i}$, $C = C_i$, $B = D_{m,i}$, and $S = S_i$ now gives

$$\mathrm{Bl}_{S_{i,N}} N_i \quad \text{Oka.} \tag{5.2}$$

Let

$$\pi_i : \mathrm{Bl}_{S_i} Y_{m,i} \longrightarrow Y_{m,i}$$

be the blow-down. Both U and N_i are analytic-Zariski open in $Y_{m,i}$: their complements are respectively C_i and $Y_{m,i} \setminus N_i$. Lemma 5.2 identifies

$$\pi_i^{-1}(U) \simeq \mathrm{Bl}_{S_U} U, \qquad \pi_i^{-1}(N_i) \simeq \mathrm{Bl}_{S_{i,N}} N_i$$

as analytic-Zariski-open subsets of $\mathrm{Bl}_{S_i} Y_{m,i}$. They are Oka by (5.1) and (5.2). Since $C_i \subset N_i$ and $Y_{m,i} = U \cup C_i$, these two subsets cover the blow-up. Kusakabe localization gives

$$\mathrm{Bl}_{S_i} Y_{m,i} \quad \text{is Oka for every } i. \tag{5.3}$$

Finally, $Y_{m,i}$ is analytic-Zariski open in Y_m, with complement $\bigcup_{j \neq i} C_j$. Lemma 5.2 identifies $\mathrm{Bl}_{S_i} Y_{m,i}$ with an analytic-Zariski-open subset of $\mathrm{Bl}_S Y_m$. The finitely many sets $Y_{m,i}$ cover Y_m, so their inverse images cover $\mathrm{Bl}_S Y_m$. By (5.3) every member of this cover is Oka. Kusakabe localization, Theorem 3.2, proves that $\mathrm{Bl}_S Y_m$ is Oka. □

Theorem 5.18. *For $m = 3, 4$ and every finite subset $Q \subset M_m$, the manifold*

$$\mathrm{Bl}_Q M_m$$

is Oka.

Proof. Let $\pi_m : \widehat{M}_m \to M_m$ be the cyclic covering and set $\widehat{Q} = \pi_m^{-1}(Q)$. Lemma 5.4 gives an unramified covering

$$\mathrm{Bl}_{\widehat{Q}} \widehat{M}_m \longrightarrow \mathrm{Bl}_Q M_m.$$

For every translated band $Y_m^{(r)}$, the core–toroidal decomposition, Lemma 4.53, gives

$$\widehat{Q} \cap Y_m^{(r)} = D_r \sqcup F_r,$$

where D_r is a finite union of full residual-deck orbits through common core points and F_r is finite. Hence

$$\mathrm{Bl}_{D_r \cup F_r} Y_m^{(r)}$$

is Oka by Theorem 5.17. By Lemma 5.2, these are precisely the inverse images of the bands in $\mathrm{Bl}_{\widehat{Q}} \widehat{M}_m$. Their complements are inverse images of the analytic complements of the bands, so they are analytic-Zariski open. They cover the lifted blow-up. Localization makes $\mathrm{Bl}_{\widehat{Q}}\,\widehat{M}_m$ Oka, and covering invariance descends the Oka property to $\mathrm{Bl}_Q\,M_m$. □

Proof of the blow-up assertion in Theorem 1.2. Recall the analytic-Zariski-open cover

$$U_1 = X \setminus S_1, \qquad U_2 = X \setminus S_2, \qquad U_1 \cup U_2 = X,$$

and the finite unramified coverings

$$\varpi_4 : M_4 \to U_1, \qquad \varpi_3 : M_3 \to U_2.$$

For $i = 1, 2$, the set $P_i = P \cap U_i$ is finite. Let $Q_4 = \varpi_4^{-1}(P_1)$ and $Q_3 = \varpi_3^{-1}(P_2)$. Theorem 5.18 makes $\mathrm{Bl}_{Q_4}\,M_4$ and $\mathrm{Bl}_{Q_3}\,M_3$ Oka. By Lemma 5.4, they are finite unramified coverings of $\mathrm{Bl}_{P_1}\,U_1$ and $\mathrm{Bl}_{P_2}\,U_2$, respectively. Covering invariance makes these two blow-ups Oka.

Let $\widetilde{X} = \mathrm{Bl}_P\,X$. Lemma 5.2 identifies the inverse images of U_1 and U_2 in $\widetilde{X}$ with $\mathrm{Bl}_{P_1}\,U_1$ and $\mathrm{Bl}_{P_2}\,U_2$. They are analytic-Zariski open and cover $\widetilde{X}$. Kusakabe localization proves that $\widetilde{X}$ is Oka. □

Corollary 5.19. *For every $a \in \mathcal{A}$ and $p \in X_a$, the blow-up $\mathrm{Bl}_p\,X_a$ is an Oka complex manifold diffeomorphic to $\mathbf{P}^3$, but it is not biholomorphic to the standard projective threefold.*

Proof. The Oka assertion is the one-point case of Theorem 1.2. Since X_a is diffeomorphic to S^6, the smooth point-blow-up is diffeomorphic to $\mathbf{P}^3$; this standard identification is recalled by Huckleberry–Kebekus–Peternell [4]. Let $E \simeq \mathbf{P}^2$ be its exceptional divisor. The blow-up canonical bundle formula gives

$$K_{\mathrm{Bl}_p X_a} \simeq \pi^* K_{X_a} \otimes \mathcal{O}_{\mathrm{Bl}_p X_a}(2E).$$

Now $H^2(X_a, \mathbf{Z}) = 0$ by Alpöge's smooth-type theorem [2, Theorem 8.1], and $[E]$ generates $H^2(\mathrm{Bl}_p\,X_a, \mathbf{Z}) \simeq \mathbf{Z}$: its restriction to E is $c_1(\mathcal{O}_E(-1))$, a generator of $H^2(E, \mathbf{Z})$. Consequently $c_1(T_{\mathrm{Bl}_p X_a}) = -2[E]$ has divisibility 2. For the standard $\mathbf{P}^3$, $c_1(T_{\mathbf{P}^3}) = 4h$ with h primitive. A biholomorphism preserves the first Chern class and its divisibility, so the two complex structures differ. □

The point-centre theorem concerns simultaneous blow-ups at distinct points. For an arbitrary smooth compact curve, the available general conclusion is the Oka–1 property. We record it in the same section because it clarifies exactly which flexibility survives without the discrete-centre arguments above.

Corollary 5.20. *If $C \subset X_a$ is a smooth compact complex curve, possibly disconnected, then $\mathrm{Bl}_C\,X_a$ is Oka–1. The same holds after any finite sequence of blow-ups along smooth centres.*

Proof. Every Oka manifold is Oka–1. The blow-up invariance theorem of Forstnerič–Lárusson [28, Theorem 3.1] applies to each smooth centre, including disconnected ones. Iterate it for the last assertion. □

For the standard exceptional curve in the local conifold model, the blow-up has the stronger algebraic ellipticity property.

Proposition 5.21. *Let $R = \mathrm{Tot}(\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1))$ and let $C \simeq \mathbf{P}^1$ be its zero section. Then*

$$\mathrm{Bl}_C\,R \simeq \mathrm{Tot}(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-1, -1)),$$

and this smooth algebraic threefold is algebraically elliptic and Oka.

Proof. For a vector bundle $E \to B$, the incidence description of the blow-up of its zero section identifies $\mathrm{Bl}_B\,E$ with the tautological line bundle on $\mathbb{P}(E)$. Here $E = \mathcal{O}_{\mathbf{P}^1}(-1) \otimes \mathbf{C}^2$, so $\mathbb{P}(E) = \mathbf{P}^1 \times \mathbf{P}^1$ and its tautological line is $\mathcal{O}(-1, -1)$. The four products of standard affine charts on the two projective-line factors trivialize this line bundle; their inverse images are copies of $\mathbb{A}^3$. A smooth algebraic threefold with such a Zariski-open cover lies in class $\mathscr{A}_0$, hence in class $\mathscr{A}$ in the terminology of Lárusson–Truong. Their class-$\mathscr{A}$ theorem [27, Corollary 2] makes it algebraically subelliptic; Kaliman–Zaidenberg's equivalence [29, Theorem 1.1] makes it algebraically elliptic. The Oka assertion follows from Gromov's theorem [1]. □

6. Logarithmic geometry and deformations

Fix an admissible parameter $a \in \mathcal{A}$, and abbreviate $(X, f, W) = (X_a, f_a, W_a)$. The finite uniformizations, period equivariance, cusp quotient, and gluing used in this and the next section were established in Section 2. No topological hypothesis on X is used.

6.1. **The root base and cusp resolution.** Set $m_1 = 3$ and $m_2 = 4$, and let σ_j denote the canonical section of $O_B(p_j)$. We use the analytic root stack

$$\mathcal{B} = \sqrt[3]{(O_B(p_1), \sigma_1)} \times_B \sqrt[4]{(O_B(p_2), \sigma_2)}, \qquad \rho : \mathcal{B} \longrightarrow B. \tag{6.1}$$

Thus an object of the j-th factor over a complex space $T \to B$ is a line bundle $\mathscr{R}$ on T, a section r, and an isomorphism $\mathscr{R}^{\otimes m_j} \simeq O_B(p_j)|_T$ carrying r^{m_j} to $\sigma_j|_T$. We write $(\mathscr{R}_j, \tau_j)$ for the universal root on $\mathcal{B}$.

Lemma 6.1. *There is an isomorphism of smooth proper Deligne–Mumford curves*

$$\mathcal{B} \simeq \mathbf{P}^{\mathrm{stk}}(3,4) := [(\mathbf{C}^2 \setminus \{0\})/\mathbf{C}^*], \qquad \lambda \cdot (x,y) = (\lambda^3 x, \lambda^4 y),$$

under which $p_1 = (y = 0)$, $p_2 = (x = 0)$, and

$$(\mathscr{R}_1, \tau_1) = (O_{\mathcal{B}}(4), y), \qquad (\mathscr{R}_2, \tau_2) = (O_{\mathcal{B}}(3), x).$$

Moreover,

$$\mathrm{Pic}(\mathcal{B}) \simeq \mathbf{Z}, \qquad O_{\mathcal{B}}(1) \text{ is a generator}, \qquad \rho^* O_B(1) = O_{\mathcal{B}}(12), \tag{6.2}$$

$$K_{\mathcal{B}} = O_{\mathcal{B}}(-7), \qquad T_{\mathcal{B}}(-p_0) = O_{\mathcal{B}}(-5). \tag{6.3}$$

If $S = \mathbf{C}[x,y]$ with $\deg x = 3$ and $\deg y = 4$, then, for every $r \in \mathbf{Z}$,

$$H^0(\mathcal{B}, O_{\mathcal{B}}(r)) = S_r, \qquad H^1(\mathcal{B}, O_{\mathcal{B}}(r)) = S^{\vee}_{-r-7}. \tag{6.4}$$

The higher coherent cohomology vanishes. In particular, $O(-1)$, $O(-2)$, $O(-5)$, and $O(-6)$ are acyclic, and $H^1(\mathcal{B}, O(1)) = 0$.

Proof. The stabilizer along $y = 0$ is μ_3, and the stabilizer along $x = 0$ is μ_4. The two displayed equivariant functions therefore provide the universal roots: $O(4)^{\otimes 3} = O(12)$ and $O(3)^{\otimes 4} = O(12)$. The complement of the origin in $\mathbf{C}^2$ has trivial algebraic Picard group, since the removed set has codimension two in the factorial affine plane. Its invertible regular functions are constants. A $\mathbf{C}^*$-linearization of the trivial bundle is therefore a character, indexed by $\mathbf{Z}$; the corresponding bundles are precisely $O(r)$. This proves the Picard assertion directly. Analytification induces an equivalence on coherent sheaves for this proper algebraic stack [12, Theorem 7.4]; in particular every analytic line bundle is algebraizable and the same Picard calculation holds for the analytic stack $\mathcal{B}$. The degree-12 invariant linear system gives the coarse map and hence (6.2). The canonical character of the quotient is the negative sum of the weights, so $K_{\mathcal{B}} = O(-3-4)$. The point p_0 is nonstacky and has class $O(12)$, which gives $T_{\mathcal{B}}(-p_0) = O(7-12)$. For cohomology, cover $\mathcal{B}$ by the quotient charts $x \neq 0$ and $y \neq 0$. Each is a quotient of an affine line by μ_3 or μ_4; since these groups are linearly reductive over $\mathbf{C}$, the charts and their intersection have no higher coherent cohomology. In degree r the resulting Čech complex is

$$S[x^{-1}]_r \oplus S[y^{-1}]_r \longrightarrow S[x^{-1}, y^{-1}]_r.$$

Its kernel is S_r, and its cokernel has basis the monomials $x^{-i} y^{-j}$ with $i, j \geq 1$ and $3i + 4j = -r$. Pairing $x^{-i} y^{-j}$ with $x^{i-1} y^{j-1}$ identifies this cokernel with $S^{\vee}_{-r-7}$ and proves (6.4). There is no higher cohomology because this is a two-chart acyclic cover. The asserted vanishings follow because none of the relevant graded pieces contains a monomial. □

Let

$$S_j = (f^{-1}(p_j))_{\mathrm{red}}.$$

The finite local model of Subsection 2.3 has coarse coordinate $t_j = s_j^{m_j}$ and a free affine cyclic action on the total space. Equivalently, the multiple-fibre identities (1.1) give the equality of effective Cartier divisors

$$f^*(p_j) = m_j S_j. \tag{6.5}$$

Lemma 6.2. *There is a morphism, unique up to a unique 2-isomorphism,*

$$\bar{f} : X \longrightarrow \mathcal{B}, \qquad \rho \circ \bar{f} = f. \tag{6.6}$$

If $\widetilde{D}_j$ is the root disc with coordinate s_j, and $\mathcal{P}_j \to \widetilde{D}_j$ is the finite torus family before taking the order-m_j affine quotient, then

$$\begin{array}{ccc} \mathcal{P}_j & \longrightarrow & X|_{D_j} \\ \big\downarrow & & \big\downarrow \bar{f} \\ \widetilde{D}_j & \longrightarrow & \mathcal{B}|_{D_j} \end{array} \tag{6.7}$$

is 2*-Cartesian. The horizontal arrows are finite etale* μ_{m_j}*-torsors. The coarse map* $s_j \mapsto s_j^{m_j}$ *is instead branched at the origin.*

Proof. Set $L_j = O_X(S_j)$ and let u_j be its canonical section. Equation (6.5) supplies an isomorphism

$$\alpha_j : L_j^{\otimes m_j} \xrightarrow{\sim} f^* O_B(p_j), \qquad \alpha_j(u_j^{m_j}) = f^*\sigma_j.$$

The universal property of the two root stacks gives (6.6) and its stated 2-categorical uniqueness.

Locally, $\mathcal{B}|_{D_j} = [\widetilde{D}_j/\mu_{m_j}]$; the other root has a nowhere-vanishing section there and hence contributes no stabilizer. The equivariant affine action on $\mathcal{P}_j$ is the free action recorded in (2.22). Consequently $[\mathcal{P}_j/\mu_{m_j}]$ is represented by the manifold $X|_{D_j}$, and the standard identity

$$[\mathcal{P}_j/\mu_{m_j}] \times_{[\widetilde{D}_j/\mu_{m_j}]} \widetilde{D}_j \simeq \mathcal{P}_j$$

proves (6.7). □

Analytic root stacks and their quotient charts are supplied by [10, Definition 4.1, Proposition 4.2, and §5]; the line-bundle-and-section universal property and its 2-Cartesian base-change form are [11, Definition 2.2.1 and Remarks 2.2.2–2.2.3]. In the argument below, representability, properness, and flatness are checked directly after pullback to the displayed analytic quotient charts, rather than being delegated to a blanket locality theorem. The direct two-chart calculation above proves the weighted quotient cohomology without invoking a separate duality result. The coherent-sheaf equivalence of [12, Theorem 7.4] transfers this calculation and the algebraic Picard group to the analytic stack used here.

Lemma 6.3. *The morphism* $\bar{f} : X \to \mathcal{B}$ *is representable, proper, and flat of relative dimension two. It is smooth over the root charts at* p_1, p_2 *and semistable, but not smooth, over* p_0*. Furthermore,*

$$\bar{f}^*(p_0) = W \tag{6.8}$$

as reduced Cartier divisors.

Proof. Use the etale atlas consisting of $B \setminus \{p_1, p_2\}$ and the two root discs $\widetilde{D}_j$. The pullback over $\widetilde{D}_j$ is $\mathcal{P}_j$ by Lemma 6.2; hence every atlas pullback is a morphism of complex spaces and $\bar{f}$ is representable. Properness is local on this atlas by the definition of properness for a representable analytic-stack morphism. Away from the two multiple points it is inherited from the proper map f of the fixed family, while $\mathcal{P}_j \to \widetilde{D}_j$ is a proper family of compact complex tori, as proved in Proposition 2.7.

The map on a finite root chart is a holomorphic submersion. The same is true on the smooth core. At the cusp, the toric local equation (2.32) gives

$$\mathbf{C}\{q\} \longrightarrow \mathbf{C}\{z_0, z_1, z_2\}, \qquad q \longmapsto z_0 z_1 z_2. \tag{6.9}$$

The target is torsion-free over the discrete valuation ring $\mathbf{C}\{q\}$ and is therefore flat. Thus every pullback to the displayed analytic atlas is flat, which is the flatness criterion for this representable morphism. The same equation shows that the central divisor is reduced, proving (6.8). □

We next examine the cusp fibre and its multiplicative resolution.

By the fixed normalization (2.48), the normalization of W is the smooth complete toric surface associated with the hexagonal fan:

$$\nu : E_0 \longrightarrow W.$$

The quotient of the periodic triangulation has one vertex, three edge orbits, and two triangle orbits. Denote the edge directions by

$$d_h = e_1, \qquad d_v = e_2, \qquad d_d = e_2 - e_1,$$

and set

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \lambda_e = B_0^{-1} d_e.$$

Thus $\lambda_h = (0, 1)$, $\lambda_v = (-1, 0)$, and $\lambda_d = (-1, -1)$. For each edge let $C_e \simeq \mathbf{P}^1$ be the corresponding boundary curve in E_0. Its two branches in the normalization are

$$\rho_e^0 : C_e \hookrightarrow E_0, \qquad \rho_e^1 = \widetilde{\Psi}_{-\lambda_e, a} \circ (C_e \hookrightarrow E_{d_e}),$$

where $\widetilde{\Psi}$ is the lifted central action of Proposition 2.20; its proof appears in Appendix B. Write P_+ and P_- for the two triple points and $x_e^\pm \in C_e$ for their preimages. The two branch maps have the same composite to the quotient; denote it by

$$j_e = \nu \circ \rho_e^0 = \nu \circ \rho_e^1 : C_e \longrightarrow W.$$

Lemma 6.4. *There is an exact sequence of sheaves of $\mathcal{O}_W$-modules*

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \nu_* \mathcal{O}_{E_0} \xrightarrow{\delta^0} \bigoplus_{e=h,v,d} j_{e*}\mathcal{O}_{C_e} \xrightarrow{\delta^1} \mathbf{C}_{P_+} \oplus \mathbf{C}_{P_-} \longrightarrow 0, \tag{6.10}$$

where

$$\delta^0(s)_e = (\rho_e^1)^* s - (\rho_e^0)^* s, \tag{6.11}$$

$$\delta^1(t_h, t_v, t_d) = \big(t_h(x_h^+) - t_v(x_v^+) + t_d(x_d^+), t_h(x_h^-) - t_v(x_v^-) + t_d(x_d^-)\big). \tag{6.12}$$

It is the normalized complex of the augmented normalization semisimplicial $\mathcal{O}_W$-algebra and therefore retains its induced multiplicative structure.

Proof. Alpöge [2, Lemma 9.12(i)] proves the normalization resolution for this periodic hexagonal cusp: one normalization component, three double curves, and two triple points. The local models are respectively smooth, $xy = 0$, and $xyz = 0$. The parameter a changes the maps ρ_e^1 by torus translations on the boundary curves, but does not change these local models or the ordered branch incidence. On each double curve the first face map is therefore the difference in (6.11). Ordering the two triangle representatives as $[0, e_1, e_2]$ and $[e_1, e_2, e_1 + e_2]$ makes both cellular boundaries $h - v + d$; this gives (6.12) and matches the sign convention of the cited lemma. Its local exactness argument applies to these same branch maps for every a. The maps are the alternating face maps of the normalization semisimplicial algebra, whose normalized complex carries the Alexander–Whitney product. □

Proposition 6.5. *One has*

$$(h^0, h^1, h^2)(W, \mathcal{O}_W) = (1, 2, 1), \tag{6.13}$$

and the cup product is a nondegenerate alternating pairing:

$$\bigwedge^2 H^1(W, \mathcal{O}_W) \xrightarrow{\sim} H^2(W, \mathcal{O}_W). \tag{6.14}$$

Proof. The complete toric surface E_0 is rational, and every C_e is a projective line. Thus these spaces have no higher structure-sheaf cohomology, and (6.10) reduces derived global sections to

$$\mathbf{C} \xrightarrow{0} \mathbf{C}^3 \xrightarrow{M} \mathbf{C}^2, \qquad M = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Its cohomology is

$$\mathbf{C}, \qquad \{(a_h, a_v, a_d) : a_h - a_v + a_d = 0\} \simeq \mathbf{C}^2, \qquad \mathbf{C}^2/\mathbf{C}(1,1) \simeq \mathbf{C},$$

which proves (6.13).

It remains to check the multiplication, since the dimensions alone do not exclude the zero pairing. The ordered normalization complex carries the Alexander–Whitney product. If $\alpha = (\alpha_h, \alpha_v, \alpha_d)$ and $\beta = (\beta_h, \beta_v, \beta_d)$ are 1-cocycles, their values on the two ordered triangles are

$$\alpha \smile \beta = (\alpha_h \beta_d, \alpha_d \beta_h).$$

The two triangles induce opposite orientations, so the identification of H^2 with $\mathbf{C}$ is $(u_+, u_-) \mapsto u_+ - u_-$. Hence

$$[\alpha] \smile [\beta] = \alpha_h \beta_d - \alpha_d \beta_h. \tag{6.15}$$

For $\alpha = (1, 1, 0)$ and $\beta = (0, 1, 1)$, the right-hand side is 1. Since both sides of (6.14) are one-dimensional, the map is an isomorphism. □

The multiplicative comparison used here is internal to the displayed resolution. The unit map from $\mathcal{O}_W$ to the normalized semisimplicial complex is a multiplicative quasi-isomorphism by Lemma 6.4. Endow its normalized cochains with the Alexander–Whitney product. Applying derived global sections to this differential graded algebra computes both the ordinary derived product on $R\Gamma(W, \mathcal{O}_W)$ and the two-triangle formula above. Thus (6.15) is the sheaf-cohomology cup product, not merely the product of the cellular dual complex.

6.2. **Logarithmic tangent bundle and direct images.** Set

$$\mathcal{T} = T_X(-\log W), \qquad \mathcal{V} = \ker\bigl(d\bar{f}^{\log} : \mathcal{T} \longrightarrow \bar{f}^*T_{\mathcal{B}}(-p_0)\bigr).$$

Proposition 6.6. *There is a rank-two vector bundle E on $\mathcal{B}$ such that*

$$\mathcal{V} \simeq \bar{f}^*E, \qquad E \simeq \mathcal{O}_{\mathcal{B}} \oplus \mathcal{O}_{\mathcal{B}}(-1). \tag{6.16}$$

The subline $\mathcal{O}_{\mathcal{B}} \subset E$ is canonically generated by the fixed vertical vector $\mathbf{u}_2 = (0,1)^t$; the displayed complementary splitting need not be canonical.

Proof. On the core, write $\xi_u = u_1\partial_{\zeta_1} + u_2\partial_{\zeta_2}$. The vertical differential of a deck transformation is

$$d_{\mathrm{vert}}g(\xi_u) = \xi_{R_g(z)u}.$$

At a finite point, the affine cyclic map has vertical linear part R_{g_j}, whereas the gluing translation depends only on the base and has vertical differential I_2. Conjugating by that translation therefore does not change R_{g_j}, which is the same linear transition as on the core. At the cusp, the relative logarithmic tangent module is freely generated by $x_1\partial_{x_1}$ and $x_2\partial_{x_2}$. For the action

$$(x,q) \longmapsto \bigl(c_{a,\lambda}(q)q^{B_0\lambda}x, q\bigr)$$

where $c_{a,\lambda}(q) = \exp(2\pi\mathrm{i}C_a(q)\lambda)$ componentwise, with C_a as in (2.30), the differential fixes the coefficients of these vertical logarithmic fields, and, writing $E_{\exp}$ for the exponential map (2.49), the punctured gluing satisfies

$$dE_{\exp}(\partial_{\zeta_i}) = 2\pi\mathrm{i}\,x_i\partial_{x_i}.$$

After this fixed constant rescaling, the core, finite, and cusp frames glue. They determine E and prove $\mathcal{V} = \bar{f}^*E$.

The period matrix and finite monodromies (2.2) and (2.12) give

$$\Omega_a\widehat{\delta} = \mathbf{u}_2, \qquad A_j\widehat{\delta} = \widehat{\delta}.$$

Period equivariance (2.4) therefore implies $R_g(z)\mathbf{u}_2 = \mathbf{u}_2$. The resulting vector is nonzero in every finite root frame and in every cusp logarithmic frame, and hence gives

$$0 \longrightarrow \mathcal{O}_{\mathcal{B}} \xrightarrow{\mathbf{u}_2} E \longrightarrow L \longrightarrow 0. \tag{6.17}$$

Comparing the period columns gives

$$R_{g_1} = \begin{pmatrix} -\tau^{-1} & 0 \\ (1-\mu)\tau^{-1} & 1 \end{pmatrix}, \qquad R_{g_2} = \begin{pmatrix} \tau^{-1} & 0 \\ -\mu\tau^{-1} & 1 \end{pmatrix}.$$

Thus the automorphy factors of L are $r_1 = -\tau^{-1}$ and $r_2 = \tau^{-1}$. The same comparison gives

$$\tau(g_1z) = \frac{\tau(z)-1}{\tau(z)}, \qquad \tau(g_2z) = -\frac{1}{\tau(z)}.$$

The classical modular discriminant therefore transforms as

$$\Delta_{\mathrm{mod}}(\tau(gz)) = r_g(z)^{-12}\Delta_{\mathrm{mod}}(\tau(z)).$$

Consequently $1/\Delta_{\mathrm{mod}}(\tau)$ is a meromorphic section of L^{12}. It has no zero or pole on the upper half-plane. At the cusp, the period decomposition (2.30) gives $\tau = s + h(q)$ with h holomorphic and $q = \exp(2\pi\mathrm{i}s)$; the product expansion gives

$$\Delta_{\mathrm{mod}}(\tau) = q\,u(q), \qquad u(0) \neq 0.$$

It follows that

$$L^{12} = \mathcal{O}_{\mathcal{B}}(-p_0) = \mathcal{O}_{\mathcal{B}}(-12).$$

The Picard group in (6.2) is torsion-free, so $L = \mathcal{O}_{\mathcal{B}}(-1)$. Finally,

$$\mathrm{Ext}^1(L,\mathcal{O}_{\mathcal{B}}) = H^1(\mathcal{B},\mathcal{O}_{\mathcal{B}}(1)) = 0$$

by (6.4); hence (6.17) splits. □

The eta product, weight-12 transformation law, nonvanishing in the upper half-plane, and simple cusp zero used in this proof are recorded in [18, §27.14, equations (27.14.12) and (27.14.16)–(27.14.18)]. Our normalization $\Delta_{\mathrm{mod}} = \eta^{24}$ differs from the DLMF discriminant by the nonzero constant $(2\pi)^{12}$.

Continue with the notation of Section 6. In particular, $\bar{f} : X \to \mathcal{B}$ is the proper flat root-stack lift, $\mathcal{T} = T_X(-\log W)$, and $\mathcal{V} = \bar{f}^*E$ with $E \simeq \mathcal{O}_{\mathcal{B}} \oplus \mathcal{O}_{\mathcal{B}}(-1)$. We first record the logarithmic sequence needed both for relative duality and for the tangent calculation.

Lemma 6.7. *Set*

$$M = T_{\mathcal{B}}(-p_0) = \mathcal{O}_{\mathcal{B}}(-5), \qquad \mathcal{H} = \bar{f}^*M.$$

Then there is an exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{T} \xrightarrow{d\bar{f}^{\log}} \mathcal{H} \longrightarrow 0. \tag{6.18}$$

Its determinant gives

$$\omega_{X/\mathcal{B}} \simeq \bar{f}^*(\det E)^{-1}. \tag{6.19}$$

Proof. On the smooth core and the two finite root charts, (6.18) is the ordinary relative tangent sequence of a submersion. In a cusp chart $q = z_0 z_1 z_2$, the logarithmic tangent bundle is generated by $z_i \partial_{z_i}$, and

$$d\bar{f}^{\log}\left(\sum_{i=0}^{2} a_i z_i \partial_{z_i}\right) = \left(\sum_{i=0}^{2} a_i\right) q\partial_q.$$

This map is onto. Its kernel consists of the two height-zero toric lattice directions, which are precisely the cusp frame of $\mathcal{V}$ constructed in Proposition 6.6. The local descriptions agree on the core–filling overlaps and prove exactness globally.

Dualizing the sequence and taking determinants gives

$$\det \mathcal{V}^\vee = K_X(W) \otimes \bar{f}^*\big(K_{\mathcal{B}}(p_0)\big)^{-1}.$$

Since $W = \bar{f}^* p_0$ by (6.8), the logarithmic divisors cancel. Using $\mathcal{V} = \bar{f}^*E$ leaves $\omega_{X/\mathcal{B}} = \bar{f}^*(\det E)^{-1}$. □

The proof uses the following pointwise form of analytic base change and the top-degree form of relative duality. Both are statements for complex spaces on an analytic étale atlas; their canonical maps descend along the atlas groupoid.

Lemma 6.8. *Let S be a smooth analytic Deligne–Mumford curve with finite inertia over* $\mathbf{C}$*, and let $p : Y \to S$ be a representable proper flat holomorphic map. Assume that $\mathcal{O}_Y$ is flat over S and that, for a fixed q, the number $h^q(Y_s, \mathcal{O}_{Y_s})$ is constant on geometric points of S. Then $R^q p_* \mathcal{O}_Y$ is locally free and the canonical map*

$$(R^q p_* \mathcal{O}_Y) \otimes k(s) \xrightarrow{\sim} H^q(Y_s, \mathcal{O}_{Y_s}) \tag{6.20}$$

is an isomorphism for every geometric point s. These isomorphisms are canonical on étale atlases, compatible on their double overlaps, and hence descend to S.

Proof. Choose an analytic étale atlas $U \to S$. The base change $p_U : Y \times_S U \to U$ is a proper map of complex spaces, U is reduced, and $\mathcal{O}_{Y \times_S U}$ is U-flat. The constancy hypothesis therefore allows one to apply [13, Chapter III, Theorem 4.12(ii), pp. 134–135]; local freeness and pointwise comparison follow. The arbitrary analytic base-change compatibility in [13, Chapter III, Theorem 4.10, pp. 132–133] identifies the two pullbacks on $U \times_S U$. The cocycle condition follows from functoriality of the canonical comparison maps, so the vector bundle and its fibre comparisons descend along the analytic groupoid. A geometric point may be lifted to the atlas, where the asserted map is exactly the complex-space comparison just cited. □

Lemma 6.9. *Let $p : Y \to S$ be a representable proper flat Gorenstein holomorphic map of relative dimension two between smooth analytic Deligne–Mumford stacks. Assume that $p_* \mathcal{O}_Y = \mathcal{O}_S$ and that the relevant direct images are locally free. Then the trace pairing induces an isomorphism*

$$(R^2 p_* \mathcal{O}_Y)^\vee \xrightarrow{\sim} p_* \omega_{Y/S}, \tag{6.21}$$

compatible with étale pullback and descent.

Proof. Again choose an analytic étale atlas $U \to S$ and write $p_U : Y_U \to U$. Relative analytic duality gives the canonical isomorphism

$$Rp_{U*}\omega_{Y_U/U}[2] \simeq R\mathcal{H}om_U(Rp_{U*}\mathcal{O}_{Y_U}, \mathcal{O}_U). \tag{6.22}$$

The relative trace and duality morphism for proper analytic maps are constructed in [14, p. 261 and pp. 264–266]; the identification of the dualizing complex of a smooth analytic space, and the invertible dualizing sheaf in the Gorenstein case, are [15, §3, Theorem 1, p. 81, and Proposition 2, p. 91]. Cancelling the pullback of ω_U in that duality gives (6.22).

Because the direct images of $\mathcal{O}_{Y_U}$ are locally free, the hyper-Ext spectral sequence has no positive Ext terms. Taking the degree corresponding to top fibre cohomology gives $(R^2p_{U*}\mathcal{O}_{Y_U})^\vee \simeq p_{U*}\omega_{Y_U/U}$. On $U \times_S U$, the two pullbacks agree: the projections are local biholomorphisms, and the relative trace, projection formula, and duality morphism are canonical and local. They therefore satisfy the groupoid cocycle condition and descend to the displayed isomorphism on S. □

Set

$$G = R^1\bar{f}_*\mathcal{O}_X.$$

Proposition 6.10. *The sheaves $R^q\bar{f}_*\mathcal{O}_X$, for $q = 0, 1, 2$, are locally free of ranks $1, 2, 1$ and satisfy geometric-point base change. They are*

$$R^0\bar{f}_*\mathcal{O}_X = \mathcal{O}_{\mathcal{B}}, \qquad R^1\bar{f}_*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{B}} \oplus \mathcal{O}_{\mathcal{B}}(-1), \qquad R^2\bar{f}_*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{B}}(-1). \tag{6.23}$$

The indicated direct-sum splitting of $R^1\bar{f}_*\mathcal{O}_X$ need not be canonical; its $[\gamma]$-subline is fixed.

Proof. The geometric fibres over the smooth locus are complex two-tori. The fibres over the two stacky points become smooth complex two-tori on the finite root charts. In all these cases their structure-sheaf cohomology has dimensions $(1, 2, 1)$. The cusp fibre has the same dimensions by (6.13). The hypotheses of Lemma 6.8 are supplied by Lemma 6.3; in particular, flatness of $\bar{f}$ makes $\mathcal{O}_X$ flat over $\mathcal{B}$. Thus the direct images are locally free with the asserted ranks and satisfy (6.20). All fibres are connected, so the unit map identifies $R^0\bar{f}_*\mathcal{O}_X$ with $\mathcal{O}_{\mathcal{B}}$.

On an etale atlas of $\mathcal{B}$, the source is smooth and each fibre is locally a Cartier divisor in the smooth total space; hence $\bar{f}$ is Gorenstein of relative dimension two. Apply Lemma 6.9, then use (6.19) and the projection formula:

$$\begin{aligned}(R^2\bar{f}_*\mathcal{O}_X)^\vee &\simeq \bar{f}_*\omega_{X/\mathcal{B}}\\ &\simeq \bar{f}_*\bar{f}^*(\det E)^{-1}\\ &\simeq (\det E)^{-1} \otimes \bar{f}_*\mathcal{O}_X \simeq (\det E)^{-1}.\end{aligned}$$

Dualizing and using (6.16) proves the R^2 formula in (6.23). It remains to identify $G = R^1\bar{f}_*\mathcal{O}_X$. Cup product defines a morphism of line bundles

$$\bigwedge^2 G \longrightarrow R^2\bar{f}_*\mathcal{O}_X. \tag{6.24}$$

By geometric-point base change, this is the usual nondegenerate cup product on every smooth torus fibre. At the cusp it is the isomorphism (6.14); at a finite stacky point the same statement follows after pulling back to its smooth root chart. Thus (6.24) is an isomorphism on every geometric point and therefore

$$\det G \simeq R^2\bar{f}_*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{B}}(-1). \tag{6.25}$$

We next construct a nowhere-zero section of G without appealing to the Kodaira–Spencer class. Recall the marked lattice

$$\Lambda = \mathbf{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta}\rangle$$

and its first coordinate character $\gamma \in \mathrm{Hom}(\Lambda, \mathbf{Z})$. The finite identities (2.21) give $\gamma \circ A_j = \gamma$. On a smooth fibre $F = \mathbf{C}^2/\Omega_a\Lambda$, constant group cocycles give a natural map from the right-hand side below to $H^1(F, \mathcal{O}_F)$:

$$H^1(F, \mathcal{O}_F) \simeq (\Lambda^* \otimes \mathbf{C})/\mathrm{rowspan}_{\mathbf{C}}\,\Omega_a. \tag{6.26}$$

Indeed, a linear function ℓ has coboundary $\lambda \mapsto \ell(\Omega_a\lambda)$. Conversely, if an entire function has constant differences under all period translations, its derivatives are periodic and descend to holomorphic functions on the compact torus; they are therefore constant, so the function is affine-linear. This proves precisely that the constant-cocycle map has kernel $\mathrm{rowspan}_{\mathbf{C}}\,\Omega_a$. Its quotient has dimension two, and it injects into the two-dimensional space $H^1(F, \mathcal{O}_F)$; hence it is an isomorphism and proves (6.26).

The class $[\gamma]$ is nonzero. If γ were a linear combination of the two period rows, evaluation on the columns $\widehat{w}$ and $\widehat{\delta}$ would force both coefficients to vanish, contradicting $\gamma(\widehat{\gamma}) = 1$. Across a finite filling, the gluing is a fibre translation and hence acts trivially on $H^1(F, \mathcal{O}_F)$, while its linear monodromy preserves γ. Thus the same class extends over both finite root charts.

For the cusp, use the splitting fixed in Subsection 2.4,

$$\Lambda_{\mathrm{tor}} = \mathbf{Z}\langle \widehat{w}, \widehat{\delta} \rangle, \qquad \overline{\Lambda} = \Lambda/\Lambda_{\mathrm{tor}} \simeq \mathbf{Z}^2.$$

Choose local sections of the cusp quotient with deck convention $s_i = \Psi_{\lambda_{ij}} s_j$, and set

$$c_{ij} = \gamma(\lambda_{ij}). \tag{6.27}$$

This is a holomorphic constant Cech 1-cocycle on the entire cusp neighbourhood. On the central fibre its values along the three oriented edges are

$$(\gamma(\lambda_h), \gamma(\lambda_v), \gamma(\lambda_d)) = (0, -1, -1). \tag{6.28}$$

The second coordinate character has values $(1, 0, -1)$, and (6.15) shows that their cup product equals 1. Hence (6.27) is nonzero on W.

This cusp class is the same as the class in (6.26), not merely another nonzero class. The character γ vanishes on Λ_{tor}, and therefore, for $\kappa \in \Lambda_{\mathrm{tor}}$ and $\bar{\lambda} \in \overline{\Lambda}$,

$$c(\kappa + \bar{\lambda}) = \gamma(\bar{\lambda}) = \gamma(\kappa + \bar{\lambda}).$$

The exponential gluing (2.49) identifies this full-lattice cocycle with (6.27) on the punctured cusp.

Finally, over a small Stein cusp disc D_0, the Leray five-term sequence is

$$0 = H^1(D_0, \mathcal{O}_{D_0}) \longrightarrow H^1(\bar{f}^{-1}(D_0), \mathcal{O}_X) \longrightarrow H^0(D_0, G) \longrightarrow H^2(D_0, \mathcal{O}_{D_0}) = 0.$$

Thus the neighbourhood cocycle gives a holomorphic section of G, and its nonzero central value shows that it does not vanish at p_0. Together with the core and finite comparisons, it gives a nowhere-zero global section

$$\mathcal{O}_{\mathcal{B}} \xrightarrow{[\gamma]} G.$$

Using (6.25), its quotient fits into

$$0 \longrightarrow \mathcal{O}_{\mathcal{B}} \xrightarrow{[\gamma]} G \longrightarrow \mathcal{O}_{\mathcal{B}}(-1) \longrightarrow 0.$$

The extension splits because $H^1(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(1)) = 0$. The splitting is not asserted to be canonical; the $[\gamma]$-subline is the fixed datum used below. We have proved (6.23). □

6.3. **Tangent cohomology from logarithmic to ordinary fields.** The projection formula, together with (6.16) and (6.23), gives

$$\begin{aligned} R^0 \bar{f}_* \mathcal{V} &= \mathcal{O} \oplus \mathcal{O}(-1), \\ R^1 \bar{f}_* \mathcal{V} &= \mathcal{O} \oplus 2\mathcal{O}(-1) \oplus \mathcal{O}(-2), \\ R^2 \bar{f}_* \mathcal{V} &= \mathcal{O}(-1) \oplus \mathcal{O}(-2), \end{aligned} \tag{6.29}$$

and

$$\begin{aligned} R^0 \bar{f}_* \mathcal{H} &= \mathcal{O}(-5), \\ R^1 \bar{f}_* \mathcal{H} &= \mathcal{O}(-5) \oplus \mathcal{O}(-6), \\ R^2 \bar{f}_* \mathcal{H} &= \mathcal{O}(-6). \end{aligned} \tag{6.30}$$

All line bundles in these displays except the displayed trivial summands are acyclic by (6.4). The weighted quotient calculation also gives vanishing in coherent degrees above one. Hence the Leray spectral sequence has only columns $p = 0, 1$ and no possible nonzero higher differential. In degree zero, the edge morphism gives $H^0(X, \mathcal{F}) = H^0(\mathcal{B}, R^0 \bar{f}_* \mathcal{F})$. For $\mathcal{F} = \mathcal{V}, \mathcal{H}$ and $n \geq 1$, it yields

$$0 \longrightarrow H^1(\mathcal{B}, R^{n-1} \bar{f}_* \mathcal{F}) \longrightarrow H^n(X, \mathcal{F}) \longrightarrow H^0(\mathcal{B}, R^n \bar{f}_* \mathcal{F}) \longrightarrow 0.$$

Consequently,

$$H^\bullet(X, \mathcal{V}) = (\mathbf{C}, \mathbf{C}, 0, 0), \qquad H^\bullet(X, \mathcal{H}) = (0, 0, 0, 0). \tag{6.31}$$

Since $H^\bullet(X, \mathcal{H}) = 0$, the long exact sequence of (6.18) gives

$$H^q(X, \mathcal{V}) \xrightarrow{\sim} H^q(X, \mathcal{T}) \quad \text{for every } q, \qquad H^\bullet(X, \mathcal{T}) = (\mathbf{C}, \mathbf{C}, 0, 0). \tag{6.32}$$

The unique class in $H^0(X,\mathcal{V})$, and hence in $H^0(X,\mathcal{T})$, is generated by the global vertical field $\mathbf{u}_2$. Under the Leray edge map, the unique degree-one class is $\mathbf{u}_2 \otimes [\gamma]$.

Regard the normalization as the composite $\nu : E_0 \to W \hookrightarrow X$, whose normal line bundle is

$$N_\nu = \mathrm{coker}(T_{E_0} \to \nu^* T_X).$$

Lemma 6.11. *There is an exact sequence*

$$0 \longrightarrow T_X(-\log W) \longrightarrow T_X \longrightarrow \nu_* N_\nu \longrightarrow 0, \tag{6.33}$$

and

$$N_\nu \simeq K_{E_0}. \tag{6.34}$$

Proof. The last arrow in (6.33) is the canonical composite

$$T_X \longrightarrow \nu_*\nu^* T_X \longrightarrow \nu_* N_\nu.$$

At a point where $r \in \{1,2,3\}$ branches of $W = (z_0 \cdots z_{r-1} = 0)$ meet, it is

$$\sum_i a_i \partial_{z_i} \longmapsto \left(a_i|_{z_i=0}\right)_{0\le i<r}.$$

The normal coefficients can be chosen independently, so this map is surjective. Its kernel consists of the vector fields preserving every branch ideal (z_i), namely $T_X(-\log W)$. This local description also shows why the quotient is the pushforward of the branchwise normal line on the normalization, rather than the single sheaf $\mathcal{O}_W(W)$.

On the dense cusp torus consider

$$\Theta = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge dq. \tag{6.35}$$

In a maximal toric chart the three ray generators form a lattice basis and $q = z_0 z_1 z_2$. The associated character matrix has determinant ± 1, so $\Theta = \pm dz_0 \wedge dz_1 \wedge dz_2$. Thus Θ extends holomorphically and without zeros across every toric boundary stratum. Under the full cusp action, $d \log x_i$ changes only by a multiple of dq or $d \log q$; wedging with dq removes every added term. Hence Θ is invariant under all $\lambda \in \overline{\Lambda}$ and descends to a nowhere-zero section of K_X on a neighbourhood of W.

The adjunction formula for the immersion $\nu : E_0 \to X$ gives

$$K_{E_0} \simeq \nu^* K_X \otimes N_\nu.$$

The descended form trivializes $\nu^* K_X$, proving (6.34). □

The hexagonal toric surface E_0 is smooth, complete, and rational. Serre duality therefore gives

$$H^0(E_0, K_{E_0}) = 0, \qquad H^1(E_0, K_{E_0}) = 0, \qquad H^2(E_0, K_{E_0}) = \mathbf{C}. \tag{6.36}$$

Since ν is finite, its pushforward is exact and $H^q(X, \nu_* K_{E_0}) = H^q(E_0, K_{E_0})$.

Theorem 6.12. *For every $a \in \mathcal{A}$,*

$$(h^0, h^1, h^2, h^3)(X_a, T_{X_a}) = (1,1,1,0).$$

Proof. Insert (6.32) and (6.36) into the long exact sequence of (6.33). Each connecting map has zero source or target, and cohomology above degree three vanishes by dimension. Exactness gives

$$H^0(X,\mathcal{T}) \xrightarrow{\sim} H^0(X,T_X), \qquad H^1(X,\mathcal{T}) \xrightarrow{\sim} H^1(X,T_X),$$
$$H^2(X,T_X) \xrightarrow{\sim} H^2(E_0,K_{E_0}), \qquad H^3(X,T_X) = 0.$$

In particular, $H^0(X,T_X)$ is generated by the image of $\mathbf{u}_2$. For clarity, the complete dimension calculation is

$\mathcal{F}$	h^0	h^1	h^2	h^3
$\mathcal{V}$	1	1	0	0
$\mathcal{H}$	0	0	0	0
$T_X(-\log W)$	1	1	0	0
$\nu_* K_{E_0}$	0	0	1	0
T_X	1	1	1	0.

Thus

$$(h^0, h^1, h^2, h^3)(X_a, T_{X_a}) = (1,1,1,0).$$

The calculation uses analytic data from the fibration. The cited base-change and duality theorems apply on complex-space étale atlases, and their canonical maps descend as described above. □

Corollary 6.13. *The two natural maps*

$$H^1(X, \mathcal{V}) \xrightarrow{\sim} H^1(X, T_X(-\log W)) \xrightarrow{\sim} H^1(X, T_X)$$

are isomorphisms. Under the Leray edge map, their common one-dimensional source is generated by $\mathbf{u}_2 \otimes [\gamma]$.

Proof. The first assertion is exactly the degree-one part of the two long exact sequences above. In (6.29), the only summand contributing to $H^1(X, \mathcal{V})$ is the trivial summand of $H^0(\mathcal{B}, R^1 \bar{f}_* \mathcal{V})$. It is the tensor product of the fixed $\mathbf{u}_2$-subline of E and the fixed $[\gamma]$-subline of G. □

6.4. **Kodaira–Spencer class and local deformations.** Fix an admissible parameter $a \in \mathcal{A}$. We compute the derivative of the actual family on the charts of Theorem 2.24. This is essential: a calculation on one smooth torus fibre would neither prove compatibility with the two finite fillings and the cusp nor identify the resulting class in the absolute tangent cohomology. No hypothesis on $H^2(X_a, \mathbf{Z})$ is used in this section.

We use the ordered lattice and the first coordinate functional

$$\Lambda = \mathbf{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta} \rangle, \qquad \gamma \in \operatorname{Hom}(\Lambda, \mathbf{Z}), \qquad \gamma(\widehat{\gamma}) = 1.$$

Set

$$\mathbf{u}_2 = (0,1)^t, \qquad V = \frac{\partial}{\partial \zeta_2}. \tag{6.37}$$

The period, monodromy, and cusp data established in Subsections 2.1 and 2.4 satisfy

$$\Omega_a \widehat{\delta} = \mathbf{u}_2, \qquad \frac{\partial \Omega_a}{\partial a} = \mathbf{u}_2 \otimes \gamma, \qquad \gamma A(g) = \gamma, \qquad R_g \mathbf{u}_2 = \mathbf{u}_2. \tag{6.38}$$

The last identity follows by applying period equivariance to $\widehat{\delta}$, which is fixed by the monodromy.

We also recall the precise cohomological output used below. Let

$$\mathcal{B} = \sqrt[3]{(B, p_1)} \times_B \sqrt[4]{(B, p_2)}, \qquad \bar{f}_a : X_a \longrightarrow \mathcal{B},$$

and set

$$\mathcal{T}_a = T_{X_a}(-\log W_a), \qquad \mathcal{V}_a = \ker\big(\mathcal{T}_a \longrightarrow \bar{f}_a^* T_{\mathcal{B}}(-p_0)\big).$$

The proof of Theorem 6.12, and in particular Corollary 6.13, gives rank-two bundles E and $G = R^1 \bar{f}_{a*} \mathcal{O}_{X_a}$ with

$$\mathcal{V}_a \simeq \bar{f}_a^* E, \qquad E \simeq \mathcal{O}_{\mathcal{B}} \oplus \mathcal{O}_{\mathcal{B}}(-1), \qquad G \simeq \mathcal{O}_{\mathcal{B}} \oplus \mathcal{O}_{\mathcal{B}}(-1), \tag{6.39}$$

and canonical isomorphisms

$$H^1(X_a, \mathcal{V}_a) \xrightarrow{\sim} H^1(X_a, \mathcal{T}_a) \xrightarrow{\sim} H^1(X_a, T_{X_a}). \tag{6.40}$$

In particular, the last space is one-dimensional.

The analytic-space base-change and relative-duality results underlying (6.39) are applied on an analytic étale atlas and then descended, as explained in the tangent-cohomology calculation; see [13, Chapter III, Theorems 4.10 and 4.12] and [14, pp. 261, 264–266]. No additional base-change statement is used here.

We use the groupoid convention

$$\kappa(\Phi) = \left.\frac{\partial \Phi_t}{\partial t}\right|_{t=0} \circ \Phi_0^{-1} \tag{6.41}$$

for an arrow obtained by replacing a with $a + t$. Thus

$$\kappa(\Phi \circ \Psi) = \kappa(\Phi) + \Phi_* \kappa(\Psi). \tag{6.42}$$

Reversing the direction of all Čech transitions multiplies every formula below by -1 and does not change the resulting nonzero class.

On the core atlas $U^\circ \times \mathbf{C}^2$, the lattice and deck arrows are

$$L_\lambda^a(z, \zeta) = (z, \zeta + \Omega_a(z)\lambda), \qquad G_g(z, \zeta) = (gz, R_g(z)\zeta).$$

Equation (6.38) gives

$$\kappa(L_\lambda) = \gamma(\lambda) V, \qquad \kappa(G_g) = 0. \tag{6.43}$$

The relation $G_g L_\lambda G_g^{-1} = L_{A(g)\lambda}$ is compatible with (6.42), because $\gamma A(g) = \gamma$ and $(G_g)_* V = V$. Hence (6.43) is a cocycle on the core groupoid, not merely a fibrewise calculation.

At the two finite points, let $m_1 = 3$, $m_2 = 4$, and use the twist vectors

$$v_1 = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}, \qquad v_2 = -\widehat{\gamma} - 3\widehat{u} + 3\widehat{w}.$$

They satisfy $A_j v_j = v_j$ and $\gamma(v_1) = 1$, $\gamma(v_2) = -1$. On the finite charts the affine generators are

$$r_j^a(z,\zeta) = \left(g_j z, R_{g_j}(z)\zeta + \Omega_a(g_j z)\frac{v_j}{m_j}\right).$$

Differentiation yields the constants

$$\kappa(r_j) = \frac{\gamma(v_j)}{m_j}V, \qquad \kappa(r_1) = \frac{1}{3}V, \qquad \kappa(r_2) = -\frac{1}{4}V. \tag{6.44}$$

The relations

$$r_j L_\lambda r_j^{-1} = L_{A_j\lambda}, \qquad r_j^{m_j} = L_{v_j}$$

give, after applying (6.42), respectively $\gamma A_j = \gamma$ and $m_j\gamma(v_j)V/m_j = \gamma(v_j)V$. Thus the constants in (6.44) also satisfy the finite group relations.

It remains to compare these constants with the core representative. On a sector on which $\log s_j$ is defined, the punctured gluing map is translation by

$$\sigma_{j,a}(z) = \frac{\log s_j(z)}{2\pi\mathrm{i}}\Omega_a(z)v_j.$$

Set

$$h_{j,a}(z,\zeta) = (z, \zeta + \sigma_{j,a}(z)).$$

Applying (6.41) to the varying arrow $h_{j,a+t}$ gives

$$\kappa(h_j) = \frac{\log s_j}{2\pi\mathrm{i}}\gamma(v_j)V. \tag{6.45}$$

A change $\log s_j \mapsto \log s_j + 2\pi\mathrm{i}n$ adds $n\gamma(v_j)V = \kappa(L_{nv_j})$, so the change is absorbed by a lattice arrow. Moreover,

$$\partial_a\sigma_{j,a}(g_j z) = R_{g_j}(z)\partial_a\sigma_{j,a}(z) - \frac{\gamma(v_j)}{m_j}\mathbf{u}_2. \tag{6.46}$$

This is exactly the derivative of the conjugacy $h_{j,a}r_j^a h_{j,a}^{-1} = G_{g_j}$. The logarithmic formula (6.45) is used only on the punctured overlap; the representative on the central finite chart is the constant in (6.44), and therefore extends holomorphically.

For the cusp, identify $\overline{\Lambda} = \Lambda/\mathbf{Z}\langle\widehat{w},\widehat{\delta}\rangle$ with $\mathbf{Z}^2$ in the basis $(\widehat{\gamma},\widehat{u})$. If $\lambda = (\lambda_1,\lambda_2)$, then $\gamma(\lambda) = \lambda_1$; write $e_2 = (0,1)^t$ for the second standard vertical vector. The cusp action on the dense torus is

$$\Psi_\lambda^a(x,q) = \left(\mathrm{e}^{2\pi\mathrm{i}C_a(q)\lambda}q^{B_0\lambda}x, q\right), \qquad C_a(q) = C_0(q) + aE_{21}.$$

With

$$V_0 = 2\pi\mathrm{i}x_2\frac{\partial}{\partial x_2}, \tag{6.47}$$

its derivative is

$$\kappa(\Psi_\lambda) = \gamma(\lambda)V_0. \tag{6.48}$$

In a maximal toric chart, if m_i are the dual characters, then

$$V_0 = 2\pi\mathrm{i}\sum_i \langle m_i, e_2\rangle z_i\frac{\partial}{\partial z_i}.$$

It follows that V_0 extends across $q = 0$, is tangent to each toric branch, and annihilates q. The toric shear is the identity on the vertical sublattice, so V_0 is invariant under the full cusp action.

Finally, the punctured cusp connector

$$E_{\exp}(z,\zeta) = \left(\mathrm{e}^{2\pi\mathrm{i}\zeta_1}, \mathrm{e}^{2\pi\mathrm{i}\zeta_2}, q(z)\right)$$

is independent of a and satisfies

$$\kappa(E_{\exp}) = 0, \qquad dE_{\exp}(V) = V_0, \qquad E_{\exp}L_\lambda = \Psi_{\overline{\lambda}}E_{\exp}. \tag{6.49}$$

It also satisfies $E_{\exp}G_{g_0} = E_{\exp}$, with $\kappa(G_{g_0}) = 0$. Thus the core and cusp representatives agree on their overlap.

At the toric boundary, V_0 need not be a local frame, so we do not introduce a coefficient by dividing by V_0. Instead, a scalar cocycle η on the generating arrows is given by

$$\eta(L_\lambda) = \gamma(\lambda), \qquad \eta(G_g) = 0, \qquad \eta(r_j) = \frac{\gamma(v_j)}{m_j},$$
$$\eta(h_j) = \frac{\log s_j}{2\pi \mathrm{i}}\gamma(v_j), \qquad \eta(\Psi_\lambda) = \gamma(\lambda), \qquad \eta(E_{\exp}) = 0.$$

The relations already checked show directly that η is well defined.

Lemma 6.14. *The formulae (6.43), (6.44), (6.45), and (6.48) determine a 1-cocycle on the chosen analytic presentation of X_a. Its canonical descent, or Čech, image is a class in $H^1(X_a, \mathcal{V}_a)$, and its image in $H^1(X_a, T_{X_a})$ is the Kodaira–Spencer class of the actual a-family.*

Proof. The core semidirect relation, both finite cyclic relations, the two finite-connector conjugacies, and the cusp equivariance were checked above on a generating set of arrows. Equation (6.42) then gives the cocycle relation for every composite arrow. Descent along the presentation gives the asserted sheaf-cohomology class. The representatives have no base component; at the cusp, the displayed toric expression for V_0 is logarithmic along $q^{-1}(0)$. Hence the cocycle is vertical and logarithmic, so it takes values in the intrinsic kernel $\mathcal{V}_a$. Differentiating the transition arrows of the parameter family gives the connecting class of

$$0 \longrightarrow T_{X_a} \longrightarrow T_{\mathfrak{X}_\Delta}|_{X_a} \longrightarrow \mathcal{O}_{X_a} \longrightarrow 0.$$

By the derivative convention (6.41), the image of our cocycle in the absolute tangent sheaf is exactly this Kodaira–Spencer class. □

The restriction of the scalar cocycle η to the lattice arrows is the character γ. Consequently, on a smooth fibre the restriction is

$$\mathbf{u}_2 \otimes [\gamma] \in H^0(F_z, T_{F_z}) \otimes H^1(F_z, \mathcal{O}_{F_z}). \tag{6.50}$$

The nonvanishing of $[\gamma]$ on smooth fibres and at the cusp was proved in Proposition 6.10, using (6.26) and (6.28).

We now make the global bridge explicit. Projection formula and (6.39) give

$$R^0 \bar{f}_{a*}\mathcal{V}_a \simeq E, \qquad R^1 \bar{f}_{a*}\mathcal{V}_a \simeq E \otimes G \simeq \mathcal{O}_{\mathcal{B}} \oplus 2\mathcal{O}_{\mathcal{B}}(-1) \oplus \mathcal{O}_{\mathcal{B}}(-2).$$

The root-stack line-bundle calculation gives $H^1(\mathcal{B}, E) = H^2(\mathcal{B}, E) = 0$ and no global sections of the three negative summands. Here the H^2-vanishing is the coherent cohomological-dimension-one statement for the tame root-stack curve proved in the preceding calculation. The five-term sequence for Leray consequently reduces to an isomorphism

$$\mathrm{edge} : H^1(X_a, \mathcal{V}_a) \xrightarrow{\sim} H^0\big(\mathcal{B}, R^1 \bar{f}_{a*}\mathcal{V}_a\big) \simeq \mathbf{C}. \tag{6.51}$$

The edge image of the cocycle in Lemma 6.14 is computed fibrewise by (6.50). Geometric-point base change identifies its restriction over the smooth locus with $\mathbf{u}_2 \otimes [\gamma]$. Since $R^1 \bar{f}_{a*}\mathcal{V}_a$ is locally free, two sections agreeing on this dense open set agree globally. The edge image is therefore precisely the section $\mathbf{u}_2 \otimes [\gamma]$ of the unique trivial summand in $E \otimes G$. Thus the global cocycle generates $H^1(X_a, \mathcal{V}_a)$.

Proposition 6.15. *The actual family has an isomorphism*

$$\mathrm{KS}_a : T_a\mathcal{A} \xrightarrow{\sim} H^1(X_a, T_{X_a}).$$

Proof. The tangent vector $\partial/\partial a$ is represented by the cocycle of Lemma 6.14. Its edge image is the nonzero generator $\mathbf{u}_2 \otimes [\gamma]$. The two maps in (6.40) carry this generator to a nonzero element of the one-dimensional space $H^1(X_a, T_{X_a})$. Since $T_a\mathcal{A}$ is also one-dimensional, the Kodaira–Spencer map is an isomorphism. □

The nonzero Kodaira–Spencer class now determines the analytic deformation germ. We use the following form of the Kuranishi theorem. For a compact complex manifold X, the analytic Kuranishi theorem provides a closed, possibly nonreduced, analytic subgerm

$$K \hookrightarrow (H^1(X, T_X), 0)$$

and a complete deformation over K such that the Kuranishi Kodaira–Spencer map identifies T_0K with $H^1(X, T_X)$. Equivalently, the defining obstruction map has zero differential at the origin. Every sufficiently small pointed analytic deformation of X admits, after shrinking, a pointed holomorphic classifying map ϕ to K, and its Kodaira–Spencer map satisfies

$$\mathrm{KS}_{\text{given}} = \mathrm{KS}_K \circ d\phi_0, \tag{6.52}$$

up to the invertible linear action induced by changing the marking of the central fibre. No uniqueness of ϕ is used.

We use this theorem in the analytic-space sense, including possibly nonreduced bases, as formulated by Douady [16, pp. 7, 10–12]. Equation (6.52) is the standard naturality of the connecting homomorphism defining the Kodaira–Spencer class.

Proposition 6.16. *The Kuranishi germ of X_a is smooth and reduced of dimension one:*

$$\mathrm{Kur}(X_a) \simeq (\mathbf{C}, 0).$$

After shrinking and a local reparameterization, the actual family is miniversal at a.

Proof. Set $K = \mathrm{Kur}(X_a)$. By Theorem 6.12, the ambient space $H^1(X_a, T_{X_a})$ is one-dimensional. Choose its coordinate t and write

$$\mathcal{O}_{K,0} = \mathbf{C}\{t\}/I. \tag{6.53}$$

Let $s = a' - a$ be the coordinate on a sufficiently small disc in the actual parameter family. Completeness supplies a classifying map

$$\phi : (\mathbf{C}, 0)_s \longrightarrow K.$$

After composing with the closed embedding of K into the t-line, the map on local rings is

$$\mathbf{C}\{t\} \longrightarrow \mathbf{C}\{s\}, \qquad t \longmapsto g(s). \tag{6.54}$$

By (6.52) and Proposition 6.15, one has $g'(0) \neq 0$. The one-dimensional holomorphic inverse function theorem therefore makes (6.54) an isomorphism after shrinking; in particular, it is injective. Since the same map factors through (6.53),

$$I \subset \ker\big(\mathbf{C}\{t\} \longrightarrow \mathbf{C}\{s\}\big) = 0.$$

Thus $I = 0$, including its nilpotent elements. Hence K is a smooth reduced line, and ϕ is locally biholomorphic. Pulling the Kuranishi family back by this biholomorphism proves miniversality of the actual family. This argument proves neither uniqueness of classifying maps nor universality. □

The proof of Theorem 6.12 already identifies $H^0(X_a, T_{X_a}) = \mathbf{C}V$, where V is the vertical field in (6.37). We now integrate this field and determine the kernel of its action.

The field V has the global flow

$$T_c[z, \zeta] = [z, \zeta + c\mathbf{u}_2], \qquad c \in \mathbf{C}. \tag{6.55}$$

It commutes with the core and finite arrows by $R_g\mathbf{u}_2 = \mathbf{u}_2$. At the cusp it becomes

$$(x_1, x_2, q) \longmapsto (x_1, \mathrm{e}^{2\pi \mathrm{i} c}x_2, q), \tag{6.56}$$

which extends over the toric boundary and commutes with the cusp shears. Thus (6.55) defines a global $\mathbf{C}$-action.

Lemma 6.17. *The kernel of $c \mapsto T_c$ is $\mathbf{Z}$.*

Proof. The restriction of T_c to the central fibre lifts through the normalization $\nu_a : E_0 \to W_a$ in (2.48). On the dense toric orbit of E_0, the lift is multiplication of the second coordinate by $\mathrm{e}^{2\pi \mathrm{i} c}$, as in (6.56). If T_c is the identity, uniqueness of the normalization lift therefore gives $\mathrm{e}^{2\pi \mathrm{i} c} = 1$, so $c \in \mathbf{Z}$. Conversely, (6.38) gives $n\mathbf{u}_2 = \Omega_a(\widehat{n\delta})$ for every $n \in \mathbf{Z}$; hence T_n is the identity on the core and, by continuation, on X_a. □

We use the following compact-complex automorphism theorem in exactly this form: if X is a compact complex manifold, then $\mathrm{Aut}(X)$ has a natural finite-dimensional complex Lie group structure, and its Lie algebra is canonically $H^0(X, T_X)$.

This is [17, Chapter III, §1, Theorem 1.1, p. 77].

Proposition 6.18. *There is an isomorphism of complex Lie groups*

$$\mathrm{Aut}^0(X_a) \simeq \mathbf{C}^*.$$

Proof. By Lemma 6.17, the flow gives an effective homomorphism

$$\mathbf{C}/\mathbf{Z} \simeq \mathbf{C}^* \longrightarrow \mathrm{Aut}^0(X_a).$$

Its differential is the nonzero vector field V. By the H^0 calculation in the proof of Theorem 6.12 and the compact-complex automorphism theorem, the connected target is one-dimensional and the differential is a Lie algebra isomorphism. The holomorphic inverse function theorem shows that the image contains a neighbourhood of the identity, hence is an open subgroup. An open subgroup of a connected topological group is the whole group, and effectiveness gives injectivity. This proves the claimed isomorphism. □

The transformations (6.55) are defined on the relative total space and cover the identity on the explicit a-base. After choosing the local identification of that base with the Kuranishi germ from Proposition 6.16, this constructed equivariant structure transports to one covering the identity of the Kuranishi base. We make no claim that miniversality alone produces a canonical action of the full stabilizer.

Proof of Theorem 1.3. The Kodaira–Spencer assertion is Proposition 6.15. The smooth, reduced one-dimensional Kuranishi germ and miniversality are Proposition 6.16. The vector-field statement follows from the proof of Theorem 6.12, and Proposition 6.18 identifies the identity component. These arguments do not establish universality and do not determine the possibly disconnected group $\mathrm{Aut}(X_a)$; the latter is determined by the global cusp argument in Subsection 7.3. □

7. Classification and deformation consequences

7.1. **Intrinsic fibration and the finite-root obstruction.**

Theorem 7.1. *For every $a \in \mathcal{A}$,*

$$K_{X_a}^{-2} \simeq f_a^* \mathcal{O}_{\mathbf{P}^1}(1), \qquad f_a^* : H^0(\mathbf{P}^1, \mathcal{O}(1)) \xrightarrow{\sim} H^0(X_a, K_{X_a}^{-2}).$$

The complete bi-anticanonical pencil recovers f_a. Every biholomorphism $\Phi : X_a \to X_{a'}$ satisfies $f_{a'} \circ \Phi = f_a$.

Proof. Fix $a \in \mathcal{A}$, and abbreviate $X = X_a$ and $f = f_a$. The relative canonical-bundle formula (6.19), together with (6.3) and (6.16), gives

$$K_X \simeq \omega_{X/\mathcal{B}} \otimes \bar{f}^* K_{\mathcal{B}} \simeq \bar{f}^*\big((\det E)^{-1} \otimes K_{\mathcal{B}}\big) \simeq \bar{f}^* \mathcal{O}_{\mathcal{B}}(-6). \tag{7.1}$$

Taking the inverse square and using (6.2) yields

$$K_X^{-2} \simeq \bar{f}^* \mathcal{O}_{\mathcal{B}}(12) \simeq \bar{f}^* \rho^* \mathcal{O}_B(1) = f^* \mathcal{O}_B(1). \tag{7.2}$$

Since $\bar{f}_* \mathcal{O}_X = \mathcal{O}_{\mathcal{B}}$, the projection formula and (6.4) give

$$\begin{aligned} H^0(X, K_X^{-2}) &\simeq H^0(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(12)) \\ &= \mathbf{C}\langle x^4, y^3 \rangle \simeq H^0(B, \mathcal{O}_B(1)). \end{aligned} \tag{7.3}$$

The two sections x^4 and y^3 have no common zero on $\mathcal{B} = \mathbf{P}^{\mathrm{stk}}(3,4)$. Hence $|K_X^{-2}|$ is base-point-free and has projective dimension one. Under the projective identification induced by (7.3), its associated morphism is $\rho \circ \bar{f} = f$. Equivalently, the zero divisor of every nonzero section of K_X^{-2} is a scheme-theoretic fibre of f, and every scheme-theoretic fibre occurs in this way.

Let now $\Phi : X_a \xrightarrow{\sim} X_{a'}$ be a biholomorphism and set

$$V_a = H^0(X_a, K_{X_a}^{-2}).$$

Functoriality of the canonical bundle gives an isomorphism $\Phi^* : V_{a'} \to V_a$. If $\kappa_a : X_a \to \mathbf{P}(V_a^\vee)$ denotes the morphism defined by the complete linear system, then

$$\kappa_{a'} \circ \Phi = \mathbf{P}\big((\Phi^*)^\vee\big) \circ \kappa_a.$$

After identifying both projective targets with B as above, there is a unique $\varphi \in \mathrm{Aut}(B)$ such that

$$f_{a'} \circ \Phi = \varphi \circ f_a. \tag{7.4}$$

Equation (7.4) identifies the scheme-theoretic fibre over b with the one over $\varphi(b)$. An automorphism of the base changes a local parameter only by a unit, so it preserves the multiplicity of the corresponding Cartier fibre. By the fibre descriptions in Theorem 2.24, p_0 is uniquely characterized by having a fibre with nonnormal reduced support. The remaining exceptional fibres are $3S_1$ and $4S_2$, and their unequal multiplicities distinguish p_1 from p_2. Thus φ fixes p_0, p_1, p_2. An automorphism of $\mathbf{P}^1$ fixing three distinct points is the identity, and hence

$$f_{a'} \circ \Phi = f_a.$$

□

Theorem 7.2. *For every $a \in \mathcal{A}$, every holomorphic map $h : X_a \to \mathbf{P}^1$ factors uniquely as $h = \varphi \circ f_a$ with a holomorphic map $\varphi : \mathbf{P}^1 \to \mathbf{P}^1$.*

Proof. Fix a and let F be a smooth torus fibre of f_a. Since X_a is diffeomorphic to S^6, $H^2(X_a, \mathbf{Z}) = 0$. Thus $c_1((h|_F)^* \mathcal{O}_{\mathbf{P}^1}(1)) = 0$. If $h|_F$ were nonconstant, the pullback of the Fubini–Study form would be semipositive and nonzero on an open subset of the Kähler surface F. Its wedge with a Kähler form on F would have strictly positive integral, contradicting the vanishing of that Chern class. Hence h is constant on every smooth fibre.

The proper holomorphic map $(f_a, h) : X_a \to \mathbf{P}^1 \times \mathbf{P}^1$ has, by Remmert's theorem, an irreducible compact analytic image Γ of dimension one. Over the complement of the three special values, Γ is the graph of a holomorphic map. The first projection $p : \Gamma \to \mathbf{P}^1$ is therefore generically one-to-one. It is also finite: a positive-dimensional fibre of a map from an irreducible compact analytic curve would be the whole curve, whereas p is surjective. Thus p is finite and bimeromorphic. On a coordinate disc of $\mathbf{P}^1$, the finite algebra $p_* \mathcal{O}_\Gamma$ sits in the meromorphic function field of the disc and is integral over its holomorphic functions. Normality of the disc forces equality, so p is an isomorphism. The second projection gives φ and $h = \varphi \circ f_a$. Surjectivity of f_a gives uniqueness. □

The μ_4-character of $\mathcal{O}_{\mathcal{B}}(-6)$ at p_2 is nontrivial. Thus the canonical-bundle formula (7.1) is naturally expressed on the root stack, while its inverse square descends from $\mathcal{O}_B(1)$ as in (7.2).

We prove the pair statement without any topological hypothesis on X_a. We use the marked core, finite-affine, and cusp data of Subsections 2.1 to 2.4. For compactness, write

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4) = (\widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta})$$

for the ordered basis of Λ, and let $E_{41}\mathbf{e}_1 = \mathbf{e}_4$ and $E_{41}\mathbf{e}_r = 0$ for $r \neq 1$. Thus $\Omega_a \mathbf{e}_4 = \mathbf{u}_2$ by (2.12). We write T_v for fibre translation by a vector v. An isomorphism over B is a biholomorphism $\Phi : X_a \to X_{a'}$ satisfying $f_{a'} \circ \Phi = f_a$.

For $j = 1, 2$, use the root coordinate and affine action of Equations (2.17) and (2.22). Denote the smooth torus family before the cyclic quotient by $\mathcal{P}_{j,a} \to \widetilde{D}_j$, write $\alpha_{j,a}$ for the displayed generator $\widetilde{g}_j$, and set $\mathcal{N}_{j,a} = \mathcal{P}_{j,a} / \langle \alpha_{j,a} \rangle$.

Lemma 7.3. *Let*

$$Z_{j,a} = \big(\mathcal{N}_{j,a} \times_{D_j} \widetilde{D}_j\big)_{\mathrm{red}}.$$

Then $\mathcal{P}_{j,a}$ is the normalization of $Z_{j,a}$. Consequently, an isomorphism of pairs over B lifts uniquely over the fixed root coordinate to an isomorphism

$$\widetilde{\Phi}_j : \mathcal{P}_{j,a} \xrightarrow{\sim}_{\widetilde{D}_j} \mathcal{P}_{j,a'},$$

and this lift satisfies

$$\widetilde{\Phi}_j \alpha_{j,a} = \alpha_{j,a'} \widetilde{\Phi}_j. \tag{7.5}$$

Its integral H_1-matrix on the central covering torus is the same matrix as on the core.

Proof. Let $p_{j,a} : \mathcal{P}_{j,a} \to \widetilde{D}_j$ be the projection. The map

$$\eta_{j,a} : \mathcal{P}_{j,a} \longrightarrow Z_{j,a}, \qquad y \longmapsto (\bar{y}, p_{j,a}(y)),$$

is proper with finite fibres, hence finite. It is surjective: after choosing a representative of a point of $\mathcal{N}_{j,a}$, a power of $\alpha_{j,a}$ gives the prescribed root coordinate. Over $s_j \neq 0$ that power is unique and the representative varies holomorphically.

On a local quotient chart near a point of the central fibre, choose a root coordinate u with coarse coordinate $t = u^{m_j}$. The base factor of the reduced fibre product has local equation

$$\big(u^{m_j} = s_j^{m_j}\big)_{\mathrm{red}} = \bigcup_{r=0}^{m_j - 1} \big(u = \mathrm{e}^{-2\pi \mathrm{i} r/m_j} s_j\big). \tag{7.6}$$

Thus the punctured locus is dense and $\eta_{j,a}$ separates precisely these branches. Since $\mathcal{P}_{j,a}$ is smooth and connected, $\eta_{j,a}$ is a finite surjective bimeromorphic map from a normal space; it is the normalization.

Now base change an isomorphism of pairs and take normalizations. Functoriality of normalization gives the unique lift $\widetilde{\Phi}_j$, and the base change fixes s_j. On $Z_{j,a}$ the transformation $(n, s_j) \mapsto (n, \mathrm{e}^{-2\pi\mathrm{i}/m_j} s_j)$ lifts to $\alpha_{j,a}$. Both sides of (7.5) lift its conjugacy by the base-changed pair map, so uniqueness proves the equality. Equivalently, on the punctured root disc a relation with $\alpha^r_{j,a'}$ would give $\mathrm{e}^{-2\pi\mathrm{i}/m_j} s_j = \mathrm{e}^{-2\pi\mathrm{i}r/m_j} s_j$, hence $r \equiv 1 \pmod{m_j}$.

Finally, the induced integral H_1-matrix is locally constant on the connected root disc. The finite-to-core gluing is a fibre translation and therefore acts trivially on H_1. The central and core matrices are consequently equal. □

The next lemma isolates the only fact from fibrewise linearization that is needed here. It applies to a pair isomorphism already known to lie over B; no intrinsic-fibration theorem and no H^2-vanishing assumption enters its proof.

Lemma 7.4. *Let $\Phi : (X_a, f_a) \xrightarrow{\sim}_B (X_{a'}, f_{a'})$ and set $k = a' - a$. Then there are $L : U^\circ \to \mathrm{GL}(2, \mathbf{C})$ and $Q \in \mathrm{GL}(4, \mathbf{Z})$ such that*

$$L(z)\Omega_a(z) = \Omega_{a'}(z)Q, \qquad QA_j = A_jQ \quad (j = 1, 2). \tag{7.7}$$

Moreover,

$$k \in \mathbf{Z}, \qquad Q = \varepsilon(I - kE_{41}), \qquad L = \varepsilon I_2, \qquad \varepsilon \in \{1, -1\}. \tag{7.8}$$

Proof. On the smooth core, translate the target by the negative of the image of the zero section. The normalized fibre map fixes zero. The affine-form lemma for compact complex tori, proved in Lemma A.12, says explicitly that a holomorphic torus map is induced by a complex-linear map followed by a translation. Hence the normalized map is induced over z by a holomorphic matrix $L(z)$. Its map on the marked integral first homology is locally constant with values in $\mathrm{GL}(4, \mathbf{Z})$; since U° is connected, it is one matrix Q. Comparing marked periods and then continuing around g_1, g_2 gives (7.7).

The explicit centralizer calculation in Appendix C gives $\{Q \in M_4(\mathbf{Q}) : QA_j = A_jQ \ (j = 1, 2)\} = \mathbf{Q}I_4 \oplus \mathbf{Q}E_{41}$. For an integral solution write $Q = rI_4 + nE_{41}$ with $r, n \in \mathbf{Z}$. If it is invertible over $\mathbf{Z}$, then $\det Q = r^4 = 1$, so $r = \varepsilon = \pm 1$. Comparing the third and fourth columns in (7.7) gives $L = \varepsilon I_2$. The first column then gives $n = -\varepsilon(a' - a)$. Thus $k \in \mathbf{Z}$ and $Q = \varepsilon(I - kE_{41})$. □

Proposition 7.5. *Every isomorphism $(X_a, f_a) \simeq_B (X_{a'}, f_{a'})$ satisfies $a' - a \in 2\mathbf{Z}$.*

Proof. Use the flat marking on the central covering torus. By Lemmas 7.3 and 7.4, the integral linear part there is the same Q as on the core. Applying the affine-form lemma for compact complex tori to this central fibre shows that the lift has the form

$$[x] \longmapsto [Qx + x_j], \qquad x_j \in \Lambda_{\mathbf{R}}/\Lambda,$$

where x_j accounts for every possible translation term, and it commutes with the same affine generator $[x] \mapsto [A_jx + v_j/m_j]$. This is equivalent to

$$(I - A_j)x_j \equiv \frac{(I - Q)v_j}{m_j} \pmod{\Lambda}. \tag{7.9}$$

Let $\mathsf{N}_j = I + A_j + \cdots + A_j^{m_j - 1}$. Applying N_j to (7.9), using $\mathsf{N}_j(I - A_j) = 0$, $A_jv_j = v_j$, and $[Q, A_j] = 0$, yields

$$(I - Q)v_j \in \mathsf{N}_j\Lambda. \tag{7.10}$$

Direct multiplication gives

$$\mathsf{N}_1 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ -12 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{pmatrix}, \qquad \mathsf{N}_2 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ -12 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \end{pmatrix}. \tag{7.11}$$

Consequently,

$$\mathsf{N}_1\Lambda = \mathbf{Z}(3, 6, -12, 0)^t + \mathbf{Z}\mathbf{e}_4, \tag{7.12}$$

$$\mathsf{N}_2\Lambda = \mathbf{Z}(4, 12, -12, 0)^t + 2\mathbf{Z}\mathbf{e}_4, \qquad \mathsf{N}_2\Lambda \cap \mathbf{Z}\mathbf{e}_4 = 2\mathbf{Z}\mathbf{e}_4. \tag{7.13}$$

Write $Q = \varepsilon(I - kE_{41})$ as in (7.8). Since $E_{41}v_1 = \mathbf{e}_4$, the first coordinate of $(I - Q)v_1$ is $1 - \varepsilon$. Membership in $\mathsf{N}_1\Lambda$ forces $3 \mid (1 - \varepsilon)$, and hence $\varepsilon = 1$. Now $Q = I - kE_{41}$ and $E_{41}v_2 = -\mathbf{e}_4$, so

$$(I - Q)v_2 = -k\mathbf{e}_4.$$

Equation (7.13) forces k to be even.

This last step cannot be recovered from the core alone. For example, $k = 1$ and $Q = I - E_{41}$ satisfy the core period and centralizer equations, but $-\mathbf{e}_4 \notin \mathsf{N}_2\Lambda$. The order-4 filling is exactly what excludes this odd shift. □

7.2. **Even shifts across the core, finite fibres, and cusp.** We now fix $k \in 2\mathbf{Z}$. Let t be the coordinate on B with $t(p_1) = 0$, $t(p_2) = 1$, and $t(p_0) = \infty$, pulled back to U.

Lemma 7.6. *There are holomorphic functions F, G on U satisfying*

$$F^3 = t, \qquad G^4 = t - 1. \tag{7.14}$$

They may be chosen so that

$$F(g_1 z) = \mathrm{e}^{-2\pi \mathrm{i}/3}F(z), \qquad F(g_2 z) = F(z), \tag{7.15}$$

$$G(g_1 z) = G(z), \qquad G(g_2 z) = \mathrm{e}^{-2\pi \mathrm{i}/4}G(z). \tag{7.16}$$

For $g_0 = (g_1 g_2)^{-1}$ one has

$$F(g_0 z) = \mathrm{e}^{2\pi \mathrm{i}/3}F(z), \qquad G(g_0 z) = \mathrm{e}^{2\pi \mathrm{i}/4}G(z). \tag{7.17}$$

The class

$$\ell = \frac{1}{2\pi \mathrm{i}} \log \frac{F}{G} \quad (\mathrm{mod}\ \mathbf{Z}) \in \mathcal{O}(U^\circ, \mathbf{C}/\mathbf{Z}) \tag{7.18}$$

satisfies

$$\ell(g_1 z) - \ell(z) = -\frac{1}{3}, \qquad \ell(g_2 z) - \ell(z) = \frac{1}{4}, \qquad \ell(g_0 z) - \ell(z) = \frac{1}{12} \quad (\mathrm{mod}\ \mathbf{Z}), \tag{7.19}$$

where $g_0 = (g_1 g_2)^{-1}$.

Proof. The zeros of t on U have orders divisible by 3, and the zeros of $t - 1$ have orders divisible by 4; neither function has a pole on U. On a simply connected Riemann surface, a nonzero holomorphic function H whose zero orders are divisible by m has a holomorphic m-th root: analytic continuation around a loop C has multiplier

$$\exp\left(\frac{2\pi \mathrm{i}}{m} \sum_p \mathrm{ind}_p(C)\, \mathrm{ord}_p(H)\right) = 1,$$

and the root extends across the zeros. This proves (7.14).

The ratios $F(gz)/F(z)$ and $G(gz)/G(z)$ extend across their zeros and are constant roots of unity. The local root coordinates at z_1, z_2 determine their values on the corresponding stabilizer. The coprimality of 3 and 4 determines the other two values, giving (7.15)–(7.16). The formulae for g_0 and for ℓ follow. We take only the $\mathbf{C}/\mathbf{Z}$-valued logarithm in (7.18); no global logarithm on U° is being chosen. □

Fix one such pair F, G for the remainder of the construction. These are new global roots on U and are distinct from the finite root coordinates s_j fixed in Subsection 2.3.

Since

$$\Omega_{a+k} = \Omega_a(I + kE_{41}), \tag{7.20}$$

the two period lattices in $\mathbf{C}^2$ are equal for integral k. The torus-valued holomorphic translation is

$$b_{a,k}(z) = -\frac{k}{12}\Omega_a(z)\mathbf{e}_1 + k\ell(z)\mathbf{u}_2. \tag{7.21}$$

Changing a local lift of ℓ by $n \in \mathbf{Z}$ changes this expression by the period $kn\mathbf{u}_2 = \Omega_a(kn\mathbf{e}_4)$, so it is well defined.

Lemma 7.7. *For $k \in 2\mathbf{Z}$, translation by $b_{a,k}$ descends to a pair isomorphism on the smooth core.*

Proof. For compatible local lifts set $D_g(z) = b_{a,k}(gz) - R_g(z)b_{a,k}(z)$. Substitution of Equations (2.2), (2.4) and (2.5) and (7.19) gives, modulo the target period lattice,

$$D_{g_1}(z) \equiv \Omega_a(g_1 z)\frac{k}{2}(\mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4), \tag{7.22}$$

$$D_{g_2}(z) \equiv \Omega_a(g_2 z)\frac{k}{2}(-\mathbf{e}_3 + \mathbf{e}_4). \tag{7.23}$$

Both classes vanish when k is even. Moreover,

$$D_{gh}(z) = D_g(hz) + R_g(hz)D_h(z). \tag{7.24}$$

Thus the two generator calculations propagate to every word in Γ; different representatives allowed by a group relation differ only by an integral period. For the cusp generator one may also calculate directly:

$$A(g_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad D_{g_0} = 0.$$

It follows that

$$\Phi^{\circ}_{a,k} : [z,\zeta]_a \longmapsto [z, \zeta + b_{a,k}(z)]_{a+k} \tag{7.25}$$

is a well-defined isomorphism over B°. $\square$

On a punctured root disc, let $\sigma_{j,a}$ be the fixed finite-to-core gluing translation in (2.25). The units

$$r_1 = \frac{F}{s_1 G}, \qquad r_2 = \frac{F s_2}{G}$$

extend without zeros across the respective central points and are invariant under g_1 and g_2. Choose invariant logarithms $\kappa_j = (2\pi\mathrm{i})^{-1}\operatorname{Log} r_j$ on $\widetilde{D}_j$. Then

$$\ell = \frac{\log s_1}{2\pi\mathrm{i}} + \kappa_1 \quad \text{near } p_1, \qquad \ell = -\frac{\log s_2}{2\pi\mathrm{i}} + \kappa_2 \quad \text{near } p_2 \quad (\mathrm{mod}\ \mathbf{Z}). \tag{7.26}$$

Indeed, the possible integer difference between a logarithm and its g_j-translate vanishes at the fixed point.

The translation in the finite chart must be

$$\begin{aligned} c_{j,a,k} &= b_{a,k} - (\sigma_{j,a+k} - \sigma_{j,a}) \\ &= -\frac{k}{12}\Omega_a \mathbf{e}_1 + k\kappa_j \mathbf{u}_2, \qquad j = 1,2. \end{aligned} \tag{7.27}$$

Here we used $(\Omega_{a+k} - \Omega_a)v_j = k\gamma(v_j)\mathbf{u}_2$ and (7.26). In particular, every logarithmic term has disappeared, so $c_{j,a,k}$ is holomorphic across the central fibre.

For later use set

$$\eta_1 = \frac{k}{2}(\mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4), \qquad \eta_2 = \frac{k}{2}(-\mathbf{e}_3 + \mathbf{e}_4). \tag{7.28}$$

A direct calculation gives

$$\begin{aligned} c_{j,a,k}(g_j z) - R_{g_j}(z)c_{j,a,k}(z) - \frac{(\Omega_{a+k} - \Omega_a)(g_j z)v_j}{m_j} \\ = \Omega_a(g_j z)\eta_j. \end{aligned} \tag{7.29}$$

For even k, $\eta_j \in \Lambda$. Thus (7.29) says that the two compositions commute exactly on the torus quotient, although their universal-cover lifts differ by the displayed integral period. Translation by $c_{j,a,k}$ therefore descends through (2.22) and extends over the entire finite filling.

The overlap compatibility is the exact identity

$$\sigma_{j,a+k} + c_{j,a,k} = \sigma_{j,a} + b_{a,k}. \tag{7.30}$$

The inverse in this chart is translation by $-c_{j,a,k}$; it too is holomorphic across the central fibre.

On the fixed cusp horodisc, use the coordinate and period decomposition of (2.30); thus $s(g_0 z) = s(z) - 1$, $q = \mathrm{e}^{2\pi\mathrm{i}s}$, and C_0 is holomorphic at $q = 0$. We recall explicitly the quotient data used for descent. Identify $\overline{\Lambda}$ with $\mathbf{Z}^2$ in the basis $(\widehat{\gamma}, \widehat{u})$, let $N \simeq \mathbf{Z}^2$ be the vertical cocharacter lattice, and set $N' = N \oplus \mathbf{Z}$. For $\lambda \in \mathbf{Z}^2$, the shear

$$\phi_\lambda : N' \longrightarrow N', \qquad (y, r) \longmapsto (y + rB_0\lambda, r) \tag{7.31}$$

preserves the fixed fan and induces a toric automorphism $\Phi^{\mathrm{tor}}_\lambda$ of the toric filling Y_ε. On the dense torus it is $\Phi^{\mathrm{tor}}_\lambda(x,q) = (q^{B_0\lambda}x, q)$. Set

$$d_{a,\lambda}(q) = \exp(2\pi \mathrm{i} C_a(q)\lambda) \in (\mathbf{C}^*)^2 \tag{7.32}$$

and

$$\Psi_{a,\lambda}(y) = d_{a,\lambda}(q(y)) \cdot \Phi^{\mathrm{tor}}_\lambda(y). \tag{7.33}$$

The maps $\Psi_{a,\lambda}$ form the fixed $\mathbf{Z}^2$-action, and the cusp filling of X_a is $Y_\varepsilon/\mathbf{Z}^2$. These formulae are part of the cusp construction of Subsection 2.4 and Proposition 2.12.

Set

$$\widehat{H} = \mathrm{e}^{2\pi \mathrm{i}s/12}\frac{F}{G}. \tag{7.34}$$

The character formulae in Lemma 7.6 make $\widehat{H}$ invariant under g_0. Moreover,

$$\widehat{H}^{12} = q\frac{t^4}{(t-1)^3} = (qt)\left(\frac{t}{t-1}\right)^3. \tag{7.35}$$

Both factors on the right are units at $q=0$, since t has a simple pole at p_0. Hence $\widehat{H}$ extends as a nonvanishing holomorphic function on the cusp disc. After taking a logarithm there, we may write

$$\ell = -\frac{s}{12} + \kappa_0(q) \pmod{\mathbf{Z}}, \qquad \kappa_0 \in \mathcal{O}(D_0). \tag{7.36}$$

The first period column is

$$\Omega_a \mathbf{e}_1 = \begin{pmatrix} 6\mu(q) \\ -s + C_{a,21}(q) \end{pmatrix}.$$

Substitution in (7.21) cancels s and gives the holomorphic vertical translation

$$\bar{b}_{a,k}(q) = \begin{pmatrix} -\dfrac{k}{2}\mu(q) \\ -\dfrac{k}{12}C_{a,21}(q) + k\kappa_0(q) \end{pmatrix}, \qquad u_{a,k}(q) = \exp(2\pi \mathrm{i}\bar{b}_{a,k}(q)) \in (\mathbf{C}^*)^2. \tag{7.37}$$

Let $M' = \operatorname{Hom}(N', \mathbf{Z})$. For a toric chart with character coordinates $z_i = \chi^{m_i}$, write $m_i = (\alpha_i, r_i) \in M'$ with $\alpha_i \in \operatorname{Hom}(N, \mathbf{Z})$. In these coordinates the translation is

$$z_i \longmapsto u_{a,k}(q)^{\alpha_i} z_i. \tag{7.38}$$

Every multiplier and its inverse are holomorphic units. The formula therefore extends across every coordinate hyperplane and is compatible on all face charts.

It remains to descend through the cusp lattice quotient. If $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2$, the source and target cusp multipliers differ by

$$d_{a+k,\lambda}(q) d_{a,\lambda}(q)^{-1} = \exp(2\pi \mathrm{i} k E_{21}\lambda) = (1, \mathrm{e}^{2\pi \mathrm{i} k\lambda_1}) = (1,1). \tag{7.39}$$

Thus they determine the same cusp action for integral k. The toric shear is the identity on the vertical lattice and preserves q, so it commutes with the q-dependent multiplication by $u_{a,k}(q)$. Hence (7.38) descends to the cusp quotient.

Finally, if $E_{\exp,a}$ denotes the fixed exponential identification on the punctured cusp, then

$$E_{\exp,a+k}(z, \zeta + b_{a,k}) = u_{a,k}(q) \cdot E_{\exp,a}(z,\zeta). \tag{7.40}$$

This is the required overlap identity. The inverse on the cusp filling is multiplication by $u_{a,k}(q)^{-1}$.

Proposition 7.8. *For $a \in \mathcal{A}$ and $k \in 2\mathbf{Z}$, translation by* (7.21)*, its two finite corrections* (7.27)*, and the cusp multiplication* (7.37) *glue to an isomorphism*

$$\Phi_{a,k} : (X_a, f_a) \xrightarrow{\sim}_B (X_{a+k}, f_{a+k}).$$

Proof. Horizontal integral translation preserves $\operatorname{Im} a$, so $a + k \in \mathcal{A}$. The three filling discs are pairwise disjoint. The finite overlaps commute by (7.30), and the cusp overlap commutes by (7.40). The local maps therefore glue. Their chartwise inverses are

$$T_{-b_{a,k}}, \qquad T_{-c_{1,a,k}}, \qquad T_{-c_{2,a,k}}, \qquad u_{a,k}^{-1},$$

and the same overlap identities show that these inverses glue and compose to the identity.

The two parameters lie in a common relatively compact disc in $\mathcal{A}$; the explicit half-plane construction in the proof of Corollary 2.28 applies to this pair. The comparison maps constructed in the proof of Proposition 2.25

(Appendix B) restrict the same core, finite, and cusp formulae and satisfy their cocycle identity. Thus the glued map induces the asserted isomorphism of the fixed intrinsic pairs, independently of those two auxiliary choices. □

Remark 7.9. Changing a logarithm used to specify ℓ or κ_j adds an integer and hence changes the corresponding translation by the period $kn\mathbf{u}_2$. Changing the logarithm used for κ_0 has the same effect before exponentiation and therefore leaves $u_{a,k}$ unchanged.

Any other pair of global roots has the form $F' = \alpha F$, $G' = \beta G$, with $\alpha \in \mu_3$ and $\beta \in \mu_4$. It changes ℓ by a constant $c \in \frac{1}{12}\mathbf{Z}/\mathbf{Z}$ and changes the global isomorphism by

$$\Phi'_{a,k} = T_{kc\mathbf{u}_2} \circ \Phi_{a,k}.$$

Thus the existence of even-shift isomorphisms and the resulting orbit relation are independent of the global roots, but an individual map is not canonical before a root normalization is chosen. No independence is claimed under changing the fixed finite uniformizers, cusp splitting, or fan marking.

Proof of the pair-equivalence assertion in Theorem 1.4. The implication from a pair isomorphism to an even parameter difference is Proposition 7.5. The converse is Proposition 7.8. The orbit-curve assertion is proved in Corollary 7.15. □

7.3. **Automorphisms, orbit classes, and deformation consequences.** In the automorphism argument, the global roots F, G, the finite uniformizers, the cusp splitting, and the fan marking are fixed. The comparison theorem of Proposition 2.25 applies to the cusp cutoff and auxiliary parameter discs.

For $c \in \mathbf{C}/\mathbf{Z}$, let T_c denote translation by $c\mathbf{u}_2$. It is an automorphism of every distinguished pair. Indeed, it is an integral-period translation on the core when c changes by an integer; it commutes with the finite affine actions because $A_j\mathbf{e}_4 = \mathbf{e}_4$; and at the cusp it is multiplication by the unit $(1, e^{2\pi i c})$. Hence these translations give a vertical action

$$\mathbf{C}/\mathbf{Z} \simeq \mathbf{C}^* \curvearrowright (X_a, f_a). \tag{7.41}$$

Its effectivity is exactly the kernel calculation in Lemma 6.17. By Theorem 1.3, this is the connected automorphism group.

We now show that there are no further components. The proof has two separate cusp steps. First, a transverse-disc lifting obstruction kills the integral class of a core translation. Second, a zero-class translation is shown to possess additive logarithms across all three fillings.

Lemma 7.10. *Every element of* $\mathrm{Aut}_B(X_a, f_a)$ *has identity linear part on the smooth torus fibres. Its translation part determines a class*

$$[c] \in H^1(\Gamma, \Lambda) \simeq \mathbf{Z}.$$

If $n \in \mathbf{Z}$ is the corresponding integer, then its residual monodromy at the cusp is

$$\bar{\lambda}_n = (0, -n) \in \overline{\Lambda} = \Lambda/\mathbf{Z}\langle \mathbf{e}_3, \mathbf{e}_4\rangle. \tag{7.42}$$

*The restrictions of this class at the order-*3 *and order-*4 *root discs are coboundaries.*

Proof. Apply Lemma 7.4 with $a' = a$. Its linear part is $Q = \varepsilon I_4$ and $L = \varepsilon I_2$. The order-3 condition (7.10), together with (7.12), gives $3 \mid (1 - \varepsilon)$ and hence $\varepsilon = 1$. The pair automorphism is therefore a fibrewise translation on the smooth core.

Pull the core and the two finite root charts back to the simply connected orbifold cover U. The translation section has a holomorphic additive lift $b : U \to \mathbf{C}^2$. We fix the target-period convention

$$b(gz) - R_g(z)b(z) = \Omega_a(gz)c(g), \qquad c(g) \in \Lambda. \tag{7.43}$$

The lattice is discrete, so $c(g)$ is constant. The cocycle identity for R_g and period equivariance give

$$c(gh) = c(g) + A(g)c(h). \tag{7.44}$$

Replacing b by $b + \Omega_a m$, for $m \in \Lambda$, replaces $c(g)$ by $c(g) + (I - A(g))m$. Thus (7.43) defines a class in $H^1(\Gamma, \Lambda)$ with this left-module convention.

Set $x = c(g_1)$ and $y = c(g_2)$. The relations $g_1^3 = g_2^4 = 1$, together with the norm matrices (7.11), give

$$x = (0, p, -2p - 3r, r)^t, \qquad y = (0, q, -q - 2t, t)^t$$

for integers p, r, q, t. If $m = (\alpha, \beta, \chi, \delta)^t$, the corresponding coboundary changes (p, r, q, t) by

$$(-6\alpha + \beta - \chi, 2\alpha - \beta, \beta + \chi, -3\alpha - \chi). \tag{7.45}$$

Consequently

$$n = p - q - 2t$$

is invariant. Conversely, taking

$$\alpha = t + q + r, \quad \beta = 2t + 2q + 3r, \quad \chi = -2t - 3q - 3r$$

in (7.45) kills r, q, t; the remaining coefficient is n. Hence $H^1(\Gamma, \Lambda) \simeq \mathbf{Z}$, with representatives

$$c_n(g_1) = n(0, 1, -2, 0)^t, \qquad c_n(g_2) = 0. \tag{7.46}$$

The first vector in (7.46) is $(I - A_1)(-\mathbf{e}_3)$, while the second is zero. Thus both finite cyclic restrictions are coboundaries. The affine twist occurs on both sides of the translation commutator and cancels, so it creates no additional finite obstruction.

Set

$$A_0 = (A_1A_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Since $g_0 = (g_1g_2)^{-1}$,

$$c_n(g_0) = -A_0 c_n(g_1) = (0, -n, n, 0)^t. \tag{7.47}$$

Moreover,

$$(I - A_0)\Lambda = \mathbf{Z}\mathbf{e}_3 \oplus \mathbf{Z}\mathbf{e}_4.$$

Therefore the image of (7.47) in $\overline{\Lambda}$ is exactly (7.42). This proves the lemma. □

The target-period convention matters at the next step. Adding the local coboundary $n(I - A_0)\mathbf{e}_2 = -n\mathbf{e}_3$ changes the raw cusp value in (7.47) to $(0, -n, 0, 0)^t$, without changing $\bar{\lambda}_n$. With $\epsilon_1 = (1, 0)^t \in \mathbf{C}^2$ and $\bar{e}_2 = (0, 1)^t \in \overline{\Lambda}$, equation (7.43) then becomes

$$b(s-1) - b(s) = n(1-s)\epsilon_1 - nC_a(q)\bar{e}_2. \tag{7.48}$$

A particular solution is

$$\frac{n}{2}s(s-1)\epsilon_1 + nsC_a(q)\bar{e}_2;$$

the remaining term is periodic. This formula records the logarithmic growth. To test whether it is compatible with extension, we restrict the automorphism to a transverse disc in the covering space. This specifies the loop and the lifted map to which the following obstruction applies.

Lemma 7.11. *Let*

$$p_a : Y_\varepsilon \longrightarrow N_a = Y_\varepsilon / \overline{\Lambda}$$

be the fixed free properly discontinuous cusp covering. Let $i : D \hookrightarrow Y_\varepsilon$ be a holomorphic embedded disc through a smooth point of the central divisor, parametrized so that $q(i(t)) = t$ and $i(D^*)$ lies in the dense torus, where $D^* = D \setminus \{0\}$. Write

$$r : \widetilde{D}^* \longrightarrow D^*, \qquad r(s) = e^{2\pi i s}, \qquad \tau(s) = s - 1$$

for its universal cover and the deck generator corresponding to a clockwise circuit about $t = 0$. Suppose a holomorphic map

$$F^\circ : N_a|_{\{q \neq 0\}} \longrightarrow N_a|_{\{q \neq 0\}}$$

over the punctured cusp disc extends to a holomorphic map $F : N_a \to N_a$ over the full cusp disc. If a holomorphic map $H^\circ : \widetilde{D}^* \to Y_\varepsilon$ satisfies

$$p_a \circ H^\circ = F^\circ \circ p_a \circ i \circ r, \qquad H^\circ \circ \tau = \Psi_{a,\lambda} \circ H^\circ,$$

then $\lambda = 0$.

Proof. Fix $s_* \in \widetilde{D}^*$. Since D is simply connected and p_a is a covering, $F \circ p_a \circ i$ has a lift $H : D \to Y_\varepsilon$ with $H(r(s_*)) = H^\circ(s_*)$. The lift is holomorphic because p_a is a local biholomorphism. Uniqueness of covering lifts on the connected space $\widetilde{D}^*$ gives $H^\circ = H \circ r$. Since $r \circ \tau = r$, it follows that $H^\circ \circ \tau = H^\circ$. The assumed monodromy relation and freeness of the $\overline{\Lambda}$-action now give $\lambda = 0$. □

We verify that the translation calculated above gives precisely such a lift on this disc. The central divisor in Y_ε is reduced with simple normal crossings, so at a general point of any component q is a local coordinate. Fixing the other local coordinates gives a disc $i : D \hookrightarrow Y_\varepsilon$ as in Lemma 7.11. On D^* write $i(t) = (x(t), t)$ in the dense torus, with $x(t) \in (\mathbf{C}^*)^2$. For the given pair automorphism F, set

$$H^\circ(s) = \big(\exp(2\pi i b(s))\, x(r(s)),\ r(s)\big). \tag{7.49}$$

The exponential identification (2.49) shows that $p_a \circ H^\circ = F^\circ \circ p_a \circ i \circ r$. Indeed, on the core F is addition by $b(s)$ in the adapted fibre coordinate. Since i is a single-valued disc in the covering space, it contributes no additional fibre deck transformation.

Set $\lambda = \bar{\lambda}_n = (0, -n)$. Equation (7.48) is equivalently

$$b(s-1) - b(s) = \big((s-1)B_0 + C_a(q)\big)\lambda.$$

Here $B_0\lambda \in \mathbf{Z}^2$, so exponentiation gives

$$\exp\big(2\pi\mathrm{i}(b(s-1) - b(s))\big) = q^{B_0\lambda}\, d_{a,\lambda}(q).$$

Together with $r(s-1) = r(s)$ and (7.33), this proves $H^\circ(s-1) = \Psi_{a,\lambda}(H^\circ(s))$ for the particular lift (7.49). The preceding lemma therefore gives $\bar{\lambda}_n = 0$, hence $n = 0$. Thus every pair automorphism has zero translation class. We now turn this vanishing into a holomorphic additive lift.

Lemma 7.12. *If a pair automorphism has zero translation class, then its translation coefficient extends to a section*

$$b \in H^0(\mathcal{B}, E).$$

Proof. Choose the additive lift on the core so that its cocycle vanishes:

$$b(gz) = R_g(z)b(z). \tag{7.50}$$

At a finite point, functoriality of the normalization after root base change from Lemma 7.3 lifts the automorphism to the smooth torus family on the root disc. Its linear part is the identity on the punctured disc and hence everywhere by the identity theorem. It is therefore translation by a holomorphic torus section. The root disc is simply connected, so this section has a holomorphic additive lift. On a punctured overlap, that lift and b differ by one fixed integral period. Adjusting by this period makes the two lifts agree and extends (7.50) through both finite root charts.

At the cusp, $R_{g_0} = I_2$ and the zero cocycle give $b(s-1) = b(s)$. Hence b descends to a single-valued holomorphic map on the punctured q-disc, and

$$u(q) = \exp(2\pi\mathrm{i}b(q)) \in (\mathbf{C}^*)^2 \tag{7.51}$$

has a global logarithm there. In particular, each component of u has winding number zero.

Because the original quotient automorphism extends across the central fibre, its composite with the covering map p_a has holomorphic local lifts near each point of that fibre. Shrink their source neighbourhoods to toric polydiscs. The complement of $q = 0$ in each such polydisc is connected, and on it a local lift and multiplication by $u(q)$ are lifts of the same quotient map. They therefore differ by one fixed deck transformation; adjust the lift by its inverse. On every intersection the adjusted lifts agree on the punctured part, because both equal multiplication by $u(q)$. That part is dense in each connected component of the intersection, so the identity theorem makes the holomorphic lifts agree throughout. They glue to a lift near $q = 0$ that restricts to multiplication by $u(q)$ on the dense torus.

For every toric character χ^α, the identity on the dense torus gives

$$\frac{\widetilde{\Phi}^*\chi^\alpha}{\chi^\alpha} = u(q)^\alpha. \tag{7.52}$$

The left-hand side is meromorphic across the toric boundary. Choose a general smooth point of a central toric component at which q is a transverse coordinate, and restrict (7.52) to a small transverse q-disc through that point. Thus $u(q)^\alpha$ is a one-variable meromorphic germ at $q = 0$. Applying this observation to a basis of the character lattice shows that both components of u are meromorphic at the origin. Write

$$u_i(q) = q^{d_i} v_i(q), \qquad d_i \in \mathbf{Z}, \qquad v_i(0) \neq 0.$$

The winding number of u_i is d_i, so the global logarithm in (7.51) gives $d_i = 0$. Each u_i therefore extends as a holomorphic unit. A logarithm on the full disc differs from the original b_i on the punctured disc by a constant integer; after that period is adjusted, b extends in the cusp logarithmic frame.

The core, finite, and cusp lifts now have exactly the transition matrices used to specify the vertical bundle E in Proposition 6.6. They therefore form a section of E on $\mathcal{B}$. □

Proposition 7.13. *For every $a \in \mathcal{A}$,*

$$\mathrm{Aut}_B(X_a, f_a) = \{T_c : c \in \mathbf{C}\}/\mathbf{Z} \simeq \mathbf{C}^*.$$

Proof. By Lemmas 7.10 and 7.11, every pair automorphism has zero translation class. By Lemma 7.12, it is induced by a section of E. Equations (6.16) and (6.4) give

$$H^0(\mathcal{B}, E) = \mathbf{C}u_2.$$

Thus the automorphism is T_c for some $c \in \mathbf{C}$. Conversely, every T_c is a pair automorphism by (7.41), and Lemma 6.17 identifies precisely the parameters differing by $\mathbf{Z}$. □

Theorem 7.14. *For every $a \in \mathcal{A}$,*

$$\mathrm{Aut}(X_a) = \mathrm{Aut}_{\mathbf{P}^1}(X_a, f_a) \simeq \mathbf{C}^*.$$

Proof. The intrinsic characterization in Theorem 7.1 makes every bare automorphism strict over B. Hence

$$\mathrm{Aut}(X_a) = \mathrm{Aut}_B(X_a, f_a) \simeq \mathbf{C}^*$$

by Proposition 7.13. The right-hand side is connected and agrees with the identity component already identified in Theorem 1.3. □

The intrinsic fibration and the pair-equivalence criterion also settle biholomorphism of the unmarked threefolds.

Proof of the bare-equivalence assertion in Theorem 1.4. If $a' - a \in 2\mathbf{Z}$, the pair isomorphism supplied by Theorem 1.4 is in particular a biholomorphism $X_a \simeq X_{a'}$. Conversely, every biholomorphism $X_a \simeq X_{a'}$ is an isomorphism of pairs over B by Theorem 7.1. The necessity assertion in Theorem 1.4 then gives $a' - a \in 2\mathbf{Z}$. □

Corollary 7.15. *The pair-isomorphism classes and the bare biholomorphism classes occurring in the family both have orbit curve*

$$\mathcal{A}/2\mathbf{Z} \simeq D^*_{M_0} := \{0 < |w| < \mathrm{e}^{-\pi M_0}\}, \qquad w = \mathrm{e}^{-\pi \mathrm{i} a}.$$

The orbit coordinate appearing in Corollary 7.15 is

$$\mathcal{A}/2\mathbf{Z} \longrightarrow D^*_{M_0}, \qquad [a] \longmapsto w = \mathrm{e}^{-\pi \mathrm{i} a}. \tag{7.53}$$

Proof of Corollary 7.15. The group $2\mathbf{Z}$ acts freely and properly discontinuously on the half-plane $\mathcal{A} = \{\operatorname{Im} a < -M_0\}$ by horizontal translation. Moreover,

$$|w| = \mathrm{e}^{\pi \operatorname{Im} a} < \mathrm{e}^{-\pi M_0}, \qquad w(a) = w(a') \Longleftrightarrow a' - a \in 2\mathbf{Z}.$$

On a simply connected open subset of $D^*_{M_0}$, a local inverse is

$$w \longmapsto \left[\frac{\mathrm{i}}{\pi} \operatorname{Log} w\right]. \tag{7.54}$$

Changing the logarithm by $2\pi \mathrm{i} m$ changes its representative by $-2m$, so the quotient class is independent of the branch. Finally, $\mathrm{d}w/\,\mathrm{d}a = -\pi \mathrm{i} w \neq 0$. These local inverses glue and prove the claim. □

By Theorem 1.4 and Corollary 7.15, this curve is the analytic orbit curve of both pair-isomorphism and bare-biholomorphism classes in the present family.

We combine the relative Oka theorem with the complete local deformation calculation. The single-fibre geometry is available at every admissible parameter by Theorem 3.11.

By Theorem 1.1, the two distinguished complements and the compact member are Oka at every admissible parameter, and every relative projection over a relatively compact parameter disc is an Oka map.

Corollary 7.16. *Let $p : \mathcal{Y} \to S$ be a proper holomorphic submersion over a complex manifold, with a marked fibre $\mathcal{Y}_{s_0} \simeq X_a$ for some $a \in \mathcal{A}$. There is a neighbourhood $S' \ni s_0$ such that $p : p^{-1}(S') \to S'$ is an Oka map.*

Proof. By Theorem 1.1, choose a small parameter disc $\Delta \Subset \mathcal{A}$ about a on which π_Δ is Oka. By Theorem 1.3, after further shrinking and local reparameterization this is a miniversal deformation of X_a. The given proper holomorphic submersion $p : \mathcal{Y} \to S$ is a deformation of its marked compact central fibre. Local completeness gives, after shrinking S to a neighbourhood S' of s_0, a holomorphic map $\varphi : S' \to \Delta$, with $\varphi(s_0) = a$, and an isomorphism of families

$$\mathcal{Y}|_{S'} \simeq S' \times_\Delta \mathfrak{X}_\Delta.$$

The classifying map is not required to be unique. By Forstnerič [23, Theorem 4.1 and the following discussion, p. 803], the pullback of an Oka map by a holomorphic map is Oka. Hence $p : \mathcal{Y}|_{S'} \to S'$ is an Oka map. □

Corollary 7.17. *Every parameter has a disc neighbourhood Δ on which π_Δ is an Oka map and distinct parameters give nonbiholomorphic fibres. The family on Δ is miniversal after shrinking and reparameterization.*

Proof. Choose $\Delta \Subset \mathcal{A}$ of diameter less than 2. If $a, a' \in \Delta$ and $X_a \simeq X_{a'}$, the exact-period theorem gives $a' - a \in 2\mathbf{Z}$. The diameter bound forces $a' = a$. The Oka and miniversality assertions follow from Theorems 1.1 and 1.3. □

The product covers in Section 3 are compatible with these distinct compact deformation classes. They remove the parameter from the lattice subgroup $\ker \gamma$; the omitted generator still acts by a parameter-dependent deck map. Passing to its quotient recovers the varying compact geometry. The orbit coordinate $w = \exp(-\pi i a)$ identifies exactly the even-integer repetitions, and the strictified groupoid in Proposition C.4 also records the constant isotropy $\mathbf{C}^*$. These are statements about the explicitly constructed family, without an identification of an ambient analytic moduli component.

Appendix A. Analytic quotient tools

This section records the analytic tools in the forms used in the proofs.

Definition A.1. A *reduced complex space* is a complex analytic space whose structure sheaf has no nonzero nilpotent sections. A reduced complex space Y is *normal* if its normalization map $\nu : Y^\nu \to Y$ is an isomorphism. Equivalently, for every $y \in Y$, the local ring $\mathcal{O}_{Y,y}$ is integrally closed in its total ring of fractions. This definition applies whether or not Y is globally irreducible.

Definition A.2. A continuous map $p : M \to S$ between Hausdorff spaces is *proper* if $p^{-1}(K)$ is compact for every compact subset $K \subset S$. A holomorphic map of complex spaces is called proper when its underlying continuous map is proper.

Definition A.3. A holomorphic map $p : Y \to Z$ of complex spaces is *finite* if it is proper and every fibre has finite underlying set.

For such a map, the Finite Mapping Theorem implies that $p_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_Z$-module; see Grauert–Remmert [7, Chapter 3, Section 1.3, pp. 64–65].

Definition A.4. Let a discrete group G act on a Hausdorff space M. The action is *properly discontinuous* if, for every pair of compact subsets $K_1, K_2 \subset M$, the set

$$\{g \in G : gK_1 \cap K_2 \neq \varnothing\}$$

is finite. The action is *free* if no nonidentity element fixes a point.

Proposition A.5. *Let M be a Hausdorff second-countable complex manifold, and let a countable discrete group G act freely and properly discontinuously on M by biholomorphisms. Then the orbit space M/G has a unique structure of a Hausdorff second-countable complex manifold for which the quotient map $q : M \to M/G$ is a local biholomorphism. If $p : M \to S$ is a G-invariant holomorphic map to a complex manifold S, then it descends uniquely to a holomorphic map $\bar{p} : M/G \to S$. If p is a submersion, then $\bar{p}$ is a submersion.*

Proof. The compact-set condition above is properness of the discrete action. Lee's holomorphic quotient theorem [8, Theorem 1.16] gives the complex manifold structure and the local biholomorphism q. The invariant map descends in quotient charts. Since $dp = d\bar{p} \circ dq$ and dq is an isomorphism, a submersion p descends to a submersion. □

Proposition A.6. *Let a discrete group G act properly discontinuously on a Hausdorff space M, let $H \leq G$, and let $V \subset M$ be an open H-invariant subset. Assume*

$$gV \cap V = \varnothing \qquad \text{for every } g \in G \setminus H. \tag{A.1}$$

Then the orbit map $M \to M/G$ induces a homeomorphism

$$V/H \xrightarrow{\cong} q_G(V)$$

onto an open subset of M/G. If M is a complex manifold, both actions are by biholomorphisms, and the actions on M and V are free, then this homeomorphism is biholomorphic for the quotient complex structures furnished by Proposition A.5.

Proof. The orbit maps are open, and condition (A.1) says that two points of V lie in the same G-orbit precisely when they lie in the same H-orbit. Thus the induced map $V/H \to q_G(V)$ is an open continuous bijection. Under the complex hypotheses it is locally the identity in the quotient charts of Proposition A.5, hence biholomorphic. □

Lemma A.7. *Let G act on a Hausdorff space M, let $q : M \to M/G$ be the quotient map, and let $p : M \to S$ be a G-invariant continuous map. Assume that M/G is Hausdorff. Suppose that for every compact $K \subset S$ there exists a compact set $C_K \subset p^{-1}(K)$ meeting every G-orbit contained in $p^{-1}(K)$. Then the descended map $\bar{p} : M/G \to S$ is proper.*

Proof. For compact $K \subset S$, every orbit over K meets C_K, so

$$\bar{p}^{-1}(K) = q(C_K).$$

The continuous image of a compact set is compact. Hence $\bar{p}^{-1}(K)$ is compact. □

A holomorphic map between reduced irreducible complex spaces is *bimeromorphic* if it restricts to a biholomorphism outside proper closed analytic subsets of source and target.

Theorem A.8. *Let $F : Y \to Z$ be a proper holomorphic map of complex analytic spaces, and let $A \subset Y$ be a closed analytic subset. Then $F(A) \subset Z$ is a closed analytic subset. In particular, if Y is compact, the image of any holomorphic map $Y \to Z$ is a compact analytic subset of Z.*

For the proper mapping theorem, applied to the restriction $F|_A$, see Grauert–Remmert [7, Chapter 10, Section 6.1, p. 213].

Theorem A.9. *Let Z be a reduced complex space. There exist a normal complex space Z^ν and a finite surjective holomorphic map $\nu_Z : Z^\nu \to Z$ with the following properties.*

(i) *Over the normal locus of Z, the map ν_Z is a biholomorphism.*
(ii) *Locally, if $Z_1, \ldots, Z_r$ are the irreducible branches of Z, then $\nu_Z^{-1}(U)$ is the disjoint union of the normalizations of those branches.*
(iii) *If Y is normal and $p : Y \to Z$ is holomorphic with no irreducible component of Y mapped into the nonnormal locus of Z, then p factors uniquely through ν_Z.*

The pair (Z^ν, ν_Z) is unique up to a unique biholomorphism over Z.

For existence, the local branch description, and the universal property, see Grauert–Remmert [7, Chapter 7, Section 2.1, p. 133; Chapter 8, Sections 3.3 and 4.1–4.3, pp. 161 and 163–165].

Theorem A.10. *Let Z be a reduced irreducible complex space, and let $\nu_Z : Z^\nu \to Z$ be its normalization. Let Y be a normal irreducible complex space, and let $p : Y \to Z$ be a finite surjective bimeromorphic holomorphic map. Then there exists a unique biholomorphism $\theta : Y \to Z^\nu$ satisfying*

$$\nu_Z \circ \theta = p.$$

Equivalently, every finite surjective bimeromorphic map from a normal irreducible complex space to Z is the normalization map, up to a unique isomorphism over Z.

For this criterion, see Grauert–Remmert [7, Chapter 8, Sections 3.3 and 4.2, pp. 161 and 164; Chapter 9, Section 3.3, pp. 179–180].

Theorem A.11. *Let $p : M \to S$ be a surjective proper C^∞ submersion between smooth paracompact manifolds. Then every point of S has an open neighbourhood V and a diffeomorphism*

$$p^{-1}(V) \cong F \times V$$

over V, where F is a smooth manifold. Thus p is a locally trivial smooth fibre bundle, and all fibres over one connected component of S are diffeomorphic.

An early compact-fibre form of the theorem, together with the horizontal-lift argument, appears in Ehresmann [5, pp. 154–155]; the stated proper-map formulation is its standard modern form.

Lemma A.12. *Let V/Λ and V'/Λ' be compact complex tori. Every holomorphic map*

$$\varphi : V/\Lambda \longrightarrow V'/\Lambda'$$

has a unique expression

$$\varphi([z]) = [Lz + b],$$

where $L : V \to V'$ is complex linear, $L(\Lambda) \subset \Lambda'$, and $b \in V'$ is defined modulo Λ'. If $\varphi(0) = 0$, then $b = 0$ modulo Λ', so φ is a homomorphism of complex Lie groups.

This affine description of maps between complex tori is standard [9, Chapter 1, §2].

Lemma A.13. *Let $f : X \to S$ be a continuous map between Hausdorff spaces. Suppose every point of S has an open neighbourhood U such that $f^{-1}(U) \to U$ is proper. Then f is proper.*

Proof. Let $K \subset S$ be compact. Choose finitely many properness neighbourhoods $U_1, \dots, U_N$ covering K. The compact Hausdorff space K is normal. For each $x \in K$, choose $i(x)$ and an open neighbourhood V_x in K with $x \in V_x \subset \overline{V}_x \subset U_{i(x)} \cap K$. Finitely many $V_{x_1}, \dots, V_{x_r}$ cover K. Each $F_j := \overline{V}_{x_j}$ is compact and lies in $U_{i(x_j)}$, so $f^{-1}(F_j)$ is compact by properness over that neighbourhood. Their finite union is $f^{-1}(K)$, proving properness. □

Appendix B. Toric and gluing verifications

This section gives the normalization, quotient, and gluing arguments used in Section 2.
The central divisor is normalized by applying these toric charts to its individual components.

Proof of Lemma 2.19. In a maximal toric chart one has $q = z_0 z_1 z_2$. At a point where exactly $r \in \{1, 2, 3\}$ components meet, the germ of D_Y is

$$\{z_0 \cdots z_{r-1} = 0\} \subset \mathbf{C}^3.$$

It is reduced. Its irreducible branches are the coordinate hyperplanes $z_i = 0$, each of which is smooth and therefore normal. By Theorem A.9, the normalization of the local ring

$$\mathbf{C}\{z_0, z_1, z_2\}/(z_0 \cdots z_{r-1})$$

is the direct sum of the r branch rings obtained by setting one z_i equal to zero. These local branch normalizations agree on overlaps because they are the restrictions of the globally defined components E_v. Every point has an affine toric neighbourhood attached to one cone, and only the finitely many rays of that cone give components through the point. Thus the local normalization has finitely many branches at every point, and the local branch normalizations glue to the disjoint union (2.46). □

The lattice action lifts to this normalization as follows.

Proof of Proposition 2.20. The restriction of Ψ_λ to D_Y carries the component E_v biholomorphically onto $E_{v+B_0\lambda}$. On the normalization this gives the lift $E_v \to E_{v+B_0\lambda}$; uniqueness of normalization lifts gives its group law. Since $B_0 \in \mathrm{GL}(2, \mathbf{Z})$, the action on component indices is simply transitive, and hence the lifted action is free. A compact subset of the disjoint union $\widetilde{D}_Y = \coprod_v E_v$ meets only finitely many components. For two compact subsets, only the finitely many translations between those component indices can carry one to meet the other. Thus the action is properly discontinuous and has the analytic quotient furnished by Proposition A.5.

Every orbit meets E_0 exactly once. Explicitly, on E_v the unique map to E_0 is $\widetilde{\Psi}_{-B_0^{-1}v,a}$; these maps are holomorphic and constant on orbits by the group law. They descend to a holomorphic inverse to the quotient map restricted to E_0. Therefore $\widetilde{D}_Y/\mathbf{Z}^2 \simeq E_0$ biholomorphically. □

We next verify that the four local pieces glue to a Hausdorff space.

Proof of Lemma 2.23. Write
$$C = \mathcal{J}_\Delta, \qquad N_j = \mathcal{N}_{j,\Delta}.$$
Let
$$O_j^C = C|_{D_j^* \times \Delta}, \qquad O_j^N = N_j|_{D_j^* \times \Delta}.$$
For uniform notation, let $\phi_j : O_j^C \to O_j^N$ be G_0 when $j = 0$ and G_j^{-1} when $j = 1, 2$. The gluing relation is generated by
$$x \sim \phi_j(x) \qquad (x \in O_j^C).$$
Because the discs D_0, D_1, D_2 are pairwise disjoint, the three open sets O_j^C are pairwise disjoint. No point of one filling piece is identified with a point of another filling piece. It follows directly that every equivalence class is either a singleton or a two-point set
$$\{x, \phi_j(x)\}$$
for a unique index j and a unique $x \in O_j^C$. In particular, an equivalence class contains at most one point from each individual piece. Hence the canonical map from every piece to the glued quotient is injective.

The image of each piece is open. For example, the saturation of the core is
$$C \sqcup O_0^N \sqcup O_1^N \sqcup O_2^N,$$
which is open in the disjoint union, and the saturation of N_j is $N_j \sqcup O_j^C$. The canonical injection of a piece is therefore a homeomorphism onto an open subset. Since the glued space is covered by four open subsets homeomorphic to second-countable spaces, the union of four countable bases is a countable basis for the glued space. The complex charts on the pieces have holomorphic transition functions because the only cross-piece transitions are the biholomorphisms ϕ_j and ϕ_j^{-1}. Thus they determine a unique complex-manifold structure once Hausdorffness has been proved.

Let $x \neq y$ be points of the glued space. The local maps to $B \times \Delta$ agree on the overlaps, so they determine a continuous map $\mathfrak{f}$. If $\mathfrak{f}(x) \neq \mathfrak{f}(y)$, choose disjoint neighbourhoods of their images in the Hausdorff base and take inverse images.

Suppose $\mathfrak{f}(x) = \mathfrak{f}(y) = (b, a)$. If $b \in D_j$, represent both points in the filling piece N_j; when $b \in D_j^*$ this is possible using ϕ_j. If b lies outside the three filling discs, represent both points in the core C. Thus there is one Hausdorff piece M containing distinct representatives $\widetilde{x}, \widetilde{y}$ of x, y.

Choose disjoint open neighbourhoods $U, V \subset M$ of $\widetilde{x}, \widetilde{y}$. Their saturations in the disjoint union are open. Explicitly, if $M = C$, then
$$\operatorname{Sat}(U) = U \sqcup \bigsqcup_j \phi_j(U \cap O_j^C),$$
and if $M = N_j$, then
$$\operatorname{Sat}(U) = U \sqcup \phi_j^{-1}(U \cap O_j^N);$$
the same formulas hold for V. These two saturations are disjoint. Indeed, an intersection point would determine one equivalence class meeting both U and V. That class would then contain two representatives in the same piece M, contrary to the one-representative property proved above and the disjointness of U, V. The quotient images of the two open saturations are therefore disjoint open neighbourhoods of x and y. Hence the glued space is Hausdorff. □

The remaining compatibility concerns the cusp cutoff and the parameter disc used in the relative construction.

Proof of Proposition 2.25. We first fix Δ and compare two radii $0 < \varepsilon' < \varepsilon$. For every valid cusp cutoff ρ for Δ, write
$$\mathcal{N}_{0,\Delta}^{(\rho)} = (Y_\rho \times \Delta)/\mathbf{Z}^2, \qquad \mathfrak{X}_\Delta^{(\rho)}$$
for the cusp quotient and the resulting global compactification, and write
$$G_0^{(\rho)} : \mathcal{J}_\Delta|_{D_\rho^* \times \Delta} \xrightarrow{\cong} \mathcal{N}_{0,\Delta}^{(\rho)}|_{D_\rho^* \times \Delta}$$
for the punctured cusp biholomorphism constructed in Proposition 2.22. All these objects are obtained by restricting the same action formula and the same exponential map. The subset $Y_{\varepsilon'} \times \Delta$ is open and invariant in $Y_\varepsilon \times \Delta$. Applying Proposition A.6 with the same group as both G and H gives an open biholomorphic embedding
$$j_{\varepsilon',\varepsilon} : \mathcal{N}_{0,\Delta}^{(\varepsilon')} \hookrightarrow \mathcal{N}_{0,\Delta}^{(\varepsilon)}. \tag{B.1}$$
Its image is the inverse image of $D_{\varepsilon'} \times \Delta$.

Let $W_\Delta^{(\varepsilon)}$ be the central divisor in the larger cusp quotient. The two open subsets

$$j_{\varepsilon',\varepsilon}(\mathcal{N}_{0,\Delta}^{(\varepsilon')}) \quad \text{and} \quad \mathcal{N}_{0,\Delta}^{(\varepsilon)} \setminus W_\Delta^{(\varepsilon)}$$

cover $\mathcal{N}_{0,\Delta}^{(\varepsilon)}$. On the first one, the map to $\mathfrak{X}_\Delta^{(\varepsilon')}$ is the inverse of (B.1), followed by the inclusion of the smaller cusp filling. On the second one, the map is given by

$$\mathcal{N}_{0,\Delta}^{(\varepsilon)} \setminus W_\Delta^{(\varepsilon)} \xrightarrow{(G_0^{(\varepsilon)})^{-1}} \mathcal{J}_\Delta|_{D_\varepsilon^*\times\Delta} \longrightarrow \mathfrak{X}_\Delta^{(\varepsilon')},$$

where the last arrow is the inclusion of the common core.

These two definitions agree on their overlap. Indeed, both punctured cusp maps are induced by the same exponential formula (2.49); restricting that formula from D_ε^* to $D_{\varepsilon'}^*$ gives

$$G_0^{(\varepsilon)}|_{D_{\varepsilon'}^*\times\Delta} = j_{\varepsilon',\varepsilon} \circ G_0^{(\varepsilon')}. \tag{B.2}$$

In the smaller glued space, a core point and its image under $G_0^{(\varepsilon')}$ represent the same point. The two local maps therefore glue to a holomorphic map from the larger cusp filling to the smaller global compactification.

On the common core and the two finite fillings take the identity. On every cusp overlap this identity agrees with the map just constructed, again by (B.2). The gluing property therefore gives a holomorphic map

$$\Theta_{\varepsilon,\varepsilon'} : \mathfrak{X}_\Delta^{(\varepsilon)} \longrightarrow \mathfrak{X}_\Delta^{(\varepsilon')}$$

over $B \times \Delta$. Reversing the roles of the two compactifications gives an inverse: on the smaller cusp filling it is the open embedding (B.1), and on the core and finite pieces it is the identity. The two composites are the identity on every member of the open gluing cover. Hence $\Theta_{\varepsilon,\varepsilon'}$ is a biholomorphism. The construction also shows directly that for $\varepsilon'' < \varepsilon' < \varepsilon$,

$$\Theta_{\varepsilon',\varepsilon''} \circ \Theta_{\varepsilon,\varepsilon'} = \Theta_{\varepsilon,\varepsilon''}.$$

This proves part (i).

Now let Δ_1 and Δ_2 be two parameter discs and let $a_0 \in \Delta_1 \cap \Delta_2$. Choose a disc $V \Subset \Delta_1 \cap \Delta_2$ containing a_0. Let the two original valid cusp cutoffs be η_1 and η_2. Restricting from Δ_i to V can only decrease the supremum in condition (V2), so each η_i remains valid for V with its original witness. Choose

$$0 < \rho < \min\{\eta_1, \eta_2\}.$$

By the monotonicity assertion of Lemma 2.11, ρ is a valid cusp cutoff for V.

Every lattice action, deck action, finite cyclic action, cusp action, and gluing map preserves the parameter. Consequently, restricting the construction over Δ_i to V gives exactly the construction obtained by applying the same formulas over V with the original cutoff η_i. In particular, the two restrictions have the same core, the same finite fillings, and the same period functions. By part (i), each is canonically biholomorphic to the construction over V made with cutoff ρ. In the notation of part (i), set the comparison from the first restriction to the second by

$$C_{12}^{V,\rho} = (\Theta_{\eta_2,\rho})^{-1} \circ \Theta_{\eta_1,\rho}.$$

If ρ' is another common valid cutoff, choose $0 < \sigma < \min\{\rho, \rho'\}$. The transitivity identity in part (i) gives

$$\Theta_{\rho,\sigma} \circ \Theta_{\eta_i,\rho} = \Theta_{\eta_i,\sigma} = \Theta_{\rho',\sigma} \circ \Theta_{\eta_i,\rho'} \qquad (i = 1, 2),$$

and hence $C_{12}^{V,\rho} = C_{12}^{V,\rho'}$. Thus the comparison is independent of the auxiliary common cutoff. If V' is another such disc, the two biholomorphisms agree on $V \cap V'$: on the common core and the two finite fillings they are both the identity, and on the cusp they are both the map determined by (B.2). They therefore patch to a canonical biholomorphism over the full overlap $\Delta_1 \cap \Delta_2$. The same explicit description makes the composite on a triple overlap equal to the direct identification: after choosing one valid cutoff smaller than the three original cutoffs, the displayed formula gives $C_{23} \circ C_{12} = C_{13}$. This proves part (ii).

Taking the fibre over a point a in part (ii) proves the final choice-independence assertion. □

Appendix C. Integral calculations and the constructed groupoid

The rational centralizer calculation follows from a sequence of explicit scalar deductions. Let

$$C_1 = QA_1 - A_1Q, \qquad C_2 = QA_2 - A_2Q, \qquad Q = (q_{rs})_{1\le r,s\le 4}.$$

Assume $C_1 = C_2 = 0$.

First use the entries

$$(C_1)_{11} = 6q_{12} - 6q_{13} - 2q_{14} = 0, \tag{C.1}$$
$$(C_1)_{12} = -q_{12} - q_{13} + q_{14} = 0, \tag{C.2}$$
$$(C_2)_{11} = -6q_{13} + 3q_{14} = 0. \tag{C.3}$$

Equation (C.3) gives $q_{14} = 2q_{13}$. Substitution into (C.2) gives $q_{12} = q_{13}$. Substitution of both relations into (C.1) gives $-4q_{13} = 0$. Hence

$$q_{12} = q_{13} = q_{14} = 0. \tag{C.4}$$

Next use

$$(C_1)_{23} = -6q_{13} + q_{22} - q_{23} - q_{33} = 0, \tag{C.5}$$
$$(C_2)_{23} = -q_{22} + q_{24} + q_{33} = 0, \tag{C.6}$$
$$(C_1)_{24} = -6q_{14} + q_{24} - q_{34} = 0, \tag{C.7}$$
$$(C_1)_{22} = -6q_{12} - q_{23} + q_{24} - q_{32} = 0, \tag{C.8}$$
$$(C_1)_{32} = 6q_{12} + q_{22} + q_{32} - q_{33} + q_{34} = 0. \tag{C.9}$$

Set $d = q_{22} - q_{33}$. Using (C.4), equations (C.5)–(C.7) give

$$q_{23} = d, \qquad q_{24} = d, \qquad q_{34} = d.$$

Equation (C.8) then gives $q_{32} = 0$, and (C.9) gives $2d = 0$. Therefore

$$q_{22} = q_{33}, \qquad q_{23} = q_{24} = q_{32} = q_{34} = 0. \tag{C.10}$$

Now use

$$(C_2)_{21} = q_{21} - 6q_{23} + 3q_{24} + q_{31} = 0, \tag{C.11}$$
$$(C_1)_{21} = -6q_{11} + q_{21} + 6q_{22} - 6q_{23} - 2q_{24} - q_{31} = 0, \tag{C.12}$$
$$(C_1)_{31} = 6q_{11} + q_{21} + 2q_{31} + 6q_{32} - 6q_{33} - 2q_{34} = 0. \tag{C.13}$$

Equation (C.11) and (C.10) give $q_{21} = -q_{31}$. Substituting this into (C.12) gives

$$q_{31} = 3(q_{22} - q_{11}),$$

whereas substitution into (C.13) gives

$$q_{31} = 6(q_{22} - q_{11}).$$

The two equalities imply $q_{22} = q_{11}$, and then

$$q_{21} = q_{31} = 0, \qquad q_{11} = q_{22} = q_{33}. \tag{C.14}$$

Finally use

$$(C_1)_{41} = 2q_{11} - q_{21} + 6q_{42} - 6q_{43} - 2q_{44} = 0, \tag{C.15}$$
$$(C_1)_{42} = 2q_{12} - q_{22} - q_{42} - q_{43} + q_{44} = 0, \tag{C.16}$$
$$(C_2)_{41} = -3q_{11} - q_{31} - 6q_{43} + 3q_{44} = 0. \tag{C.17}$$

Set $e = q_{44} - q_{11}$. By (C.4) and (C.14), equations (C.16) and (C.17) become

$$e = q_{42} + q_{43}, \qquad e = 2q_{43}.$$

Thus $q_{42} = q_{43}$. Equation (C.15) then reduces to $-2e = 0$. Consequently

$$q_{44} = q_{11}, \qquad q_{42} = q_{43} = 0. \tag{C.18}$$

Equations (C.4), (C.10), (C.14) and (C.18) show that every simultaneous rational solution is

$$Q = rI_4 + nE_{41},$$

where $r = q_{11}$ and $n = q_{41}$ are unrestricted. Conversely,

$$E_{41}A_1 = A_1E_{41} = E_{41}, \qquad E_{41}A_2 = A_2E_{41} = E_{41},$$

as one verifies by multiplying the displayed matrices in (2.2). Hence every $rI_4 + nE_{41}$ commutes with both A_1 and A_2. This proves that the common centralizer over $\mathbf{Q}$ is exactly

$$\mathbf{Q}I_4 \oplus \mathbf{Q}E_{41}.$$

The same integral marking enters the explicit even-shift isomorphisms. We examine their composition before constructing the action groupoid.

For $k \in 2\mathbf{Z}$, let $\Phi_{a,k}$ be the isomorphism constructed in Proposition 7.8. These raw maps do not yet form an action.

Proposition C.1. *For $k, l \in 2\mathbf{Z}$,*

$$\Phi_{a+k,l} \circ \Phi_{a,k} = T_{-kl/12} \circ \Phi_{a,k+l}, \tag{C.19}$$

$$\Phi_{a+k,-k} = T_{k^2/12} \circ \Phi_{a,k}^{-1}. \tag{C.20}$$

In particular, the raw backward shift is generally not the chartwise inverse constructed in Proposition 7.8.

Proof. Choose a relatively compact disc $\Delta \Subset \mathcal{A}$ containing $a, a+k, a+k+l$ and a valid cusp cutoff for it. Such a disc exists by the same half-plane calculation as in the proof of Corollary 2.28: choose a real centre far enough below the boundary so that all three points lie in a disc whose radius is smaller than the distance from its centre to the boundary. We first compute all three maps in that single relative construction. The comparison cocycle of Proposition 2.25 then transports the resulting equality to the fixed representatives. Equation (7.20) gives

$$\Omega_{a+k}\mathbf{e}_1 = \Omega_a\mathbf{e}_1 + k\mathbf{u}_2.$$

Substitution in (7.21) yields

$$b_{a,k} + b_{a+k,l} = b_{a,k+l} - \frac{kl}{12}\mathbf{u}_2. \tag{C.21}$$

The finite and cusp formulae differ from the core formula by compatible extensions of the same vertical translations, so (C.21) gives (C.19) on every chart. Taking $l = -k$ gives (C.20). The defining formula also gives $\Phi_{a,0} = \mathrm{id}$. □

The corrected translations and maps are

$$\widetilde{b}_{a,k} = b_{a,k} - \frac{k^2}{24}\mathbf{u}_2, \qquad \widetilde{\Phi}_{a,k} = T_{-k^2/24} \circ \Phi_{a,k}. \tag{C.22}$$

On the finite charts this replaces $c_{j,a,k}$ by $c_{j,a,k} - k^2\mathbf{u}_2/24$; on the cusp chart it replaces $u_{a,k}$ by $(1, \mathrm{e}^{-2\pi \mathrm{i} k^2/24})u_{a,k}$. The overlap identities of Subsection 7.2 are therefore unchanged.

Proposition C.2. *The corrected maps satisfy*

$$\widetilde{\Phi}_{a+k,l} \circ \widetilde{\Phi}_{a,k} = \widetilde{\Phi}_{a,k+l}, \qquad \widetilde{\Phi}_{a+k,-k} = \widetilde{\Phi}_{a,k}^{-1}. \tag{C.23}$$

Thus, for the fixed roots and analytic markings, they determine a strict $2\mathbf{Z}$-action on the family of distinguished pairs.

Proof. Vertical translations commute with the raw shifts. Combining (C.19) with

$$-\frac{k^2}{24} - \frac{l^2}{24} - \frac{kl}{12} = -\frac{(k+l)^2}{24}$$

gives the first identity in (C.23); the second is its specialization to $l = -k$. □

Remark C.3. If $F' = \alpha F$ and $G' = \beta G$, with $\alpha \in \mu_3$ and $\beta \in \mu_4$, then

$$\ell' = \ell + c, \qquad c \in \frac{1}{12}\mathbf{Z}/\mathbf{Z}, \qquad \widetilde{\Phi}'_{a,k} = T_{kc} \circ \widetilde{\Phi}_{a,k}.$$

The map $k \mapsto T_{kc}$ is a character of $2\mathbf{Z}$, so the new maps remain strict and give a different splitting. The strict action is attached to the chosen roots, finite uniformizers, cusp splitting, and fan marking.

Let

$$G = (\mathbf{C}/\mathbf{Z}) \times 2\mathbf{Z} \simeq \mathbf{C}^* \times 2\mathbf{Z}, \qquad (c,k) \cdot a = a + k.$$

The first factor acts trivially on $\mathcal{A}$. On the fixed family, the arrow labelled by (c, k) is

$$T_c \circ \widetilde{\Phi}_{a,k} : (X_a, f_a) \longrightarrow (X_{a+k}, f_{a+k}). \tag{C.24}$$

Vertical translations commute with the strict shifts, and (C.23) is the composition law. For each fixed k, the core, finite, and cusp formulae depend jointly holomorphically on a on every common parameter disc; the comparison cocycle preserves these maps on overlaps. The arrows therefore form an analytic family, and we obtain the constructed analytic action groupoid

$$(\mathbf{C}^* \times 2\mathbf{Z}) \ltimes \mathcal{A} \rightrightarrows \mathcal{A}. \tag{C.25}$$

Proposition C.4. *After the fixed analytic choices and quadratic strictification, the constructed action groupoid has quotient*

$$[\mathcal{A}/(\mathbf{C}^* \times 2\mathbf{Z})] \simeq D^*_{M_0} \times B\mathbf{C}^*.$$

Its arrows exhaust the biholomorphisms between every pair of geometric fibres.

Proof. The $\mathbf{C}^*$-factor acts trivially on the parameter and commutes with the $2\mathbf{Z}$-factor, giving the first product decomposition. The $2\mathbf{Z}$-action is free and properly discontinuous, so its analytic quotient stack is represented by the quotient curve. The orbit-coordinate calculation above therefore gives

$$[\mathcal{A}/(\mathbf{C}^* \times 2\mathbf{Z})] \simeq [\mathcal{A}/2\mathbf{Z}] \times B\mathbf{C}^* \simeq D^*_{M_0} \times B\mathbf{C}^*. \tag{C.26}$$

It remains to prove the assertion about geometric arrows. If $\Psi : X_a \to X_{a'}$ is a biholomorphism, then Theorem 1.4 gives $a' = a + k$ for some $k \in 2\mathbf{Z}$. The composite

$$\Psi \circ \widetilde{\Phi}_{a,k}^{-1} \in \mathrm{Aut}(X_{a+k})$$

is T_c for a unique $c \in \mathbf{C}/\mathbf{Z}$ by Theorem 7.14. Hence

$$\Psi = T_c \circ \widetilde{\Phi}_{a,k},$$

which is exactly an arrow of (C.24). The converse is part of the construction. □

The quotient in (C.26) is the quotient of the explicit action in (C.25); the $\mathbf{C}^*$ factor records its isotropy on geometric fibres.

Appendix D. The source complex structure on the six-sphere

The complex structure used throughout this paper originates in Alpöge's construction [2, Construction 6.1 and Theorem 8.1]. Its underlying idea is to complete a family of complex two-tori over a three-punctured projective line in three different ways. The finite monodromies at two punctures lead to smooth multiple fibres; the unipotent monodromy at the third leads to a reduced, nonnormal toric fibre. The choices of the three completions also determine the integral topology of the total space. We describe how these parts fit together, using the notation of Section 2.

Take $B = \mathbf{P}^1_t$ with $p_1 = 0$, $p_2 = 1$, $p_0 = \infty$, and $B^\circ = B \setminus \{p_0, p_1, p_2\}$. Alpöge [2, Proposition 2.11] uses the hyperbolic orbifold of signature $(0; 3, 4, \infty)$ on B. Its upper-half-plane uniformization U has group

$$\Gamma = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1 \rangle, \qquad g_0 = (g_1 g_2)^{-1},$$

where g_j fixes a point over p_j and g_0 is the primitive parabolic element at p_0. Removing the two finite-stabilizer orbits gives a covering $U^\circ \to B^\circ$. Thus the base simultaneously supplies an order-three point, an order-four point, and a cusp; each has a distinct local completion of the torus family.

The integral lattice $\Lambda = \mathbf{Z}\langle \widehat{\gamma}, \widehat{u}, \widehat{w}, \widehat{\delta} \rangle$ will be the first homology of a smooth fibre. The representation $g_j \mapsto A_j$ in (2.2) has $A_1^3 = A_2^4 = I$, whereas $A_0 = (A_1 A_2)^{-1}$ is unipotent. The first-coordinate character $\gamma : \Lambda \to \mathbf{Z}$ is fixed by both finite monodromies; their coinvariants leave precisely the circle measured by γ. At the cusp, the primitive rank-two lattice

$$\Lambda_{\mathrm{tor}} = \ker(A_0 - I) = \mathrm{im}(A_0 - I) = \mathbf{Z}\langle \widehat{w}, \widehat{\delta} \rangle$$

is the lattice that will be exponentiated. On $\overline{\Lambda} = \Lambda/\Lambda_{\mathrm{tor}}$, the induced map to Λ_{tor} has matrix

$$B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the bases $(\overline{\widehat{\gamma}}, \overline{\widehat{u}})$ and $(\widehat{w}, \widehat{\delta})$. Its unimodularity is what makes the cusp's periodic toric construction yield a single irreducible central fibre. These integral features are the link between the complex-geometric fillings and Alpöge's later topology calculation.

Alpöge [2, Theorem 3.4] realizes this monodromy by holomorphic period functions τ, μ, β on U. The first is a modular parameter satisfying $j(\tau) = 1728t \circ \pi$; its transformations under g_1, g_2 are $\tau \mapsto (\tau - 1)/\tau$ and

$\tau \mapsto -1/\tau$. The branching of j at 0 and 1728 matches the order-three and order-four stabilizers: at the latter point the projective modular action has order two, while its integral lift has order four. The remaining two functions solve the affine descent equations forced by the same integral monodromy. In the present notation the resulting matrix is

$$\Omega_a(z) = \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta_0(z) + a & \mu(z) & 0 & 1 \end{pmatrix}, \qquad a \in \mathscr{A},$$

where the additive freedom in β is recorded by a. The inequality $\mathscr{D}_a < 0$ from (2.6) gives $\det_{\mathbf{R}} \Omega_a = \Im\tau\, \mathscr{D}_a \neq 0$. Hence its columns form a full lattice in $\mathbf{C}^2$. The equivariance $\Omega_a(gz) = R_g(z)\Omega_a(z)A(g)^{-1}$ makes $\mathbf{C}^2/\Omega_a(z)\Lambda$ descend from U° to a proper holomorphic torus family $J_a \to B^\circ$. In particular, the period functions provide both the complex structure of each fibre and the coordinate changes needed to carry it around the base.

At p_j, a local uniformizing coordinate s_j satisfies $s_j(g_jz) = e^{-2\pi\mathrm{i}/m_j} s_j(z)$, where $(m_1, m_2) = (3, 4)$; the coarse coordinate is $t_j = s_j^{m_j}$. The pulled-back family extends over $s_j = 0$ with a smooth torus fibre. Alpöge's logarithmic transform [2, Theorem 5.4] divides this extension by the affine cyclic action

$$(z, \zeta) \longmapsto \left(g_j z, R_{g_j}(z)\zeta + \Omega_a(g_j z)\frac{v_j}{m_j}\right),$$

using the invariant vectors

$$v_1 = \widehat{\gamma} + 2\widehat{u} - 4\widehat{w}, \qquad v_2 = -\widehat{\gamma} - 3\widehat{u} + 3\widehat{w}.$$

Here $A_j v_j = v_j$, so the m_jth power of the affine lift translates by the integral period v_j and is the identity on the torus. Moreover $\gamma(v_1) = 1$ and $\gamma(v_2) = -1$. If a nonidentity power fixed a point of the central torus, application of the invariant character γ to its fixed-point equation would force $k\gamma(v_j)/m_j \in \mathbf{Z}$ for some $0 < k < m_j$, which is impossible. The quotient is therefore smooth; its reduced central surface S_j is a free cyclic quotient of a complex two-torus and is bielliptic. Because the map to the coarse base is $t_j = s_j^{m_j}$, its fibre is $m_j S_j$, namely $3S_1$ or $4S_2$. Over $s_j \neq 0$, translation by

$$\frac{\log s_j}{2\pi\mathrm{i}}\, \Omega_a(z) v_j$$

conjugates this affine action to the original monodromy action. Changing the branch of $\log s_j$ adds an integral period, so the identification with J_a over the punctured disc is single-valued.

The cusp requires a different operation. On a simply connected horodisc above p_0, choose a logarithmic coordinate s with $q = e^{2\pi\mathrm{i}s} = 1/t$ and $s(g_0 z) = s(z) - 1$. The period block Z_a formed by the first two columns of Ω_a has the form

$$Z_a = sB_0 + C_a(q), \qquad C_a \in \mathcal{O}(D_0, M_2(\mathbf{C})),$$

because $\tau - s$, μ, and $\beta_0 + a + \tau$ extend across $q = 0$. Quotienting first by the last two periods, those of Λ_{tor}, turns the fibre coordinates into $(\mathbf{C}^*)^2$. For $\bar{\lambda} \in \overline{\Lambda}$, the remaining period acts there by multiplication by

$$q^{B_0\bar{\lambda}} \exp\bigl(2\pi\mathrm{i} C_a(q)\bar{\lambda}\bigr).$$

The singular factor is thus an integral monomial in q, while the second factor is a holomorphic unit. This is the passage from the analytic period family to Alpöge's toric filling [2, Lemma 4.2 and Theorem 4.5].

More explicitly, triangulate $\mathbf{R}^2$ periodically by the two unimodular triangles in each unit square and cone the triangles from the origin in $\mathbf{R}^2 \oplus \mathbf{R}$ at height one. The resulting countable fan is smooth. Its toric threefold has a holomorphic height character q, and on each maximal affine chart this map reads $q = z_0 z_1 z_2$. Hence the central divisor upstairs is reduced with simple normal crossings. Translation of the fan by $B_0\bar{\lambda}$, together with the holomorphic unit in the preceding display, extends the residual $\overline{\Lambda}$-action over $q = 0$. Alpöge proves that, on a sufficiently small cusp disc, this action is free and properly discontinuous and its quotient is proper over the disc. Since $B_0\overline{\Lambda} = \mathbf{Z}^2$, all vertices of the periodic fan lie in one orbit. The quotient central fibre W is therefore irreducible and nonnormal, although it has normal-crossing singularities. Its normalization is the degree-six del Pezzo surface: the six boundary curves form a hexagon, and opposite sides are identified in W, producing three double curves and two triple points [2, Proposition 4.6]. Exponentiating the fibre coordinates identifies the toric quotient over $q \neq 0$ with the original family J_a over the punctured cusp disc [2, Proposition 4.7].

The three punctured identifications attach the two logarithmic fillings and the toric cusp filling to the same smooth torus family. Alpöge [2, Construction 6.1 and Theorem 6.2] shows that the resulting space is a compact Hausdorff complex threefold X^{A} with a proper map to $\mathbf{P}^1$ and exactly the fibres described above.

The affine translations also enter its topology. In Alpöge's notation the cusp gluing is untranslated, and the character values are $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$; consequently

$$12\ell_0 - 4\ell_1 - 3\ell_2 = -1.$$

Alpöge [2, Theorem 7.17, Corollary 7.29, and Theorem 8.1] proves that these choices make X^{A} simply connected with the integral homology of S^6, and then identifies its underlying smooth manifold with S^6. Transporting the complex atlas along that diffeomorphism gives the complex structure on the standard smooth six-sphere.

Our family uses precisely the monodromy matrices, finite twists, period normalization, and untranslated cusp gluing just described. The constant a varies the invariant period $\beta_0 + a$ through the admissible half-plane; the member $a = a_{\mathrm{A}}$ is Alpöge's compact threefold by Lemma 2.29. The jointly holomorphic fillings give a proper submersion over that half-plane, so its fibres are all diffeomorphic to S^6 by Corollary 2.30. The normalization of W and its paired boundary curves enter the cusp resolution and cup-product calculation in Section 6.

Motivation and chronology

Motivation. For some time, S.-Y. Xie had an idea for proving that S^6 admits no complex structure. Suppose, for contradiction, that S^6 admitted a complex structure. The resulting complex manifold would then be very close to being Oka; using the scarcity of Oka manifolds, one would try to show that it was not Oka, thereby obtaining a contradiction. This earlier idea did not succeed as originally formulated.

During the 32nd International Conference on Finite or Infinite Dimensional Complex Analysis and Applications, held in Quanzhou on 22–25 August 2026, Xie mentioned this earlier unsuccessful idea in a conversation with Yuta Kusakabe. Kusakabe instead suggested considering whether S^6—more precisely, Alpöge's complex six-sphere—is Oka. This suggestion initiated our collaboration.

Chronology and related work. By 27 August 2026, we had obtained the Oka result for the complex six-sphere, and S.-Y. Xie informed F. Forstnerič of it by email on that date. Our other Oka results, including the finite-point deletion and blow-up results, were also obtained by the end of August 2026. The present manuscript was written before we learned of Chen's preprint and was being polished when we became aware of it. Chen's preprint, submitted on 22 September 2026 [22], independently establishes the Oka property for the admissible complex six-spheres. The results here additionally address the relative Oka map, finite-point deletions and blow-ups, deformations, and exact biholomorphism classes. The finite-point arguments use complete lifted centres, character-adapted coverings, and localization on the corrected incidence charts.

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AI Use Disclosure

The research questions, mathematical insights, and principal constructions originated with the authors. Generative AI was used to assist with concrete technical problems, calculations, consistency checks, and the drafting and polishing of the exposition. The authors reviewed the AI-assisted material and take responsibility for the content of this paper.

References

[1] M. Gromov, *Oka's principle for holomorphic sections of elliptic bundles*, J. Amer. Math. Soc. **2** (1989), no. 4, 851–897, https://doi.org/10.1090/S0894-0347-1989-1001851-9. ↑ 2, 31, 35, 73

[2] L. Alpöge, *The* $(3, 4, \infty)$ *modular family of* 2*-tori, completed at its three special points, is a complex structure on* S^6, manuscript, 2026, https://alpo.ge/s6.pdf. ↑ 2, 4, 5, 12, 14, 21, 22, 73, 76, 103, 104, 105

[3] G. Etesi, *The complex structure on the six dimensional sphere*, arXiv:1509.02300v6, 2024, https://arxiv.org/abs/1509.02300. ↑ 105

[4] A. T. Huckleberry, S. Kebekus, and T. Peternell, *Group actions on S^6 and complex structures on $\mathbb{P}_3(\mathbb{C})$*, Duke Math. J. **102** (2000), no. 1, 101–124, https://doi.org/10.1215/S0012-7094-00-10214-1. ↑ 73

[5] C. Ehresmann, *Les connexions infinitésimales dans un espace fibré différentiable*, Séminaire Bourbaki, vol. 1, exposé no. 24, pp. 153–168, Société mathématique de France, 1952, https://www.numdam.org/item/SB_1948-1951__1__153_0/. ↑ 98

[6] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, 1993. ↑ 12

[7] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Grundlehren der mathematischen Wissenschaften, vol. 265, Springer, Berlin, 1984. ↑ 96, 97

[8] J. M. Lee, *Introduction to Complex Manifolds*, Graduate Studies in Mathematics, vol. 244, American Mathematical Society, Providence, 2024, https://bookstore.ams.org/gsm-244. ↑ 96

[9] H. Lange and C. Birkenhake, *Complex Abelian Varieties*, Grundlehren der mathematischen Wissenschaften, vol. 302, Springer, Berlin, 1992, https://doi.org/10.1007/978-3-662-02788-2. ↑ 98

[10] M. Talpo and A. Vistoli, *A general formalism for logarithmic structures*, Bollettino dell'Unione Matematica Italiana **11** (2018), 489–502; §§4–5, https://doi.org/10.1007/s40574-017-0149-6. ↑ 75

[11] C. Cadman, *Using stacks to impose tangency conditions on curves*, American Journal of Mathematics **129** (2007), 405–427, https://doi.org/10.1353/ajm.2007.0007. ↑ 75

[12] M. Porta and T. Y. Yu, *Higher analytic stacks and GAGA theorems*, Advances in Mathematics **302** (2016), 351–409; Theorems 7.1 and 7.4, https://doi.org/10.1016/j.aim.2016.07.017. ↑ 74, 75

[13] C. Bănică and O. Stănăşilă, *Algebraic Methods in the Global Theory of Complex Spaces*, Editura Academiei and John Wiley & Sons, Bucharest and London, 1976; Chapter III, Theorems 4.10 and 4.12, pp. 132–135. ↑ 78, 82

[14] J.-P. Ramis, G. Ruget, and J.-L. Verdier, *Dualité relative en géométrie analytique complexe*, Inventiones Mathematicae **13** (1971), 261–283; Introduction and §2, https://doi.org/10.1007/BF01406078. ↑ 79, 82

[15] J.-P. Ramis and G. Ruget, *Complexe dualisant et théorèmes de dualité en géométrie analytique complexe*, Publications Mathématiques de l'IHÉS **38** (1970), 77–91; §3, Theorem 1 and Proposition 2, https://doi.org/10.1007/BF02684652. ↑ 79

[16] A. Douady, *Le problème des modules pour les variétés analytiques complexes (d'après Masatake Kuranishi)*, Séminaire Bourbaki, exp. 277, vol. 9, 1964–1966, pp. 7–13, https://www.numdam.org/item/SB_1964-1966__9__7_0/. ↑ 85

[17] S. Kobayashi, *Transformation Groups in Differential Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 70, Springer, Berlin, 1972; Chapter III, §1, Theorem 1.1, p. 77. ↑ 85

[18] NIST Digital Library of Mathematical Functions, *Unrestricted Partitions*, §27.14, equations (27.14.12) and (27.14.16)–(27.14.18), https://dlmf.nist.gov/27.14. ↑ 78

[19] G. T. Buzzard and S. S.-Y. Lu, *Algebraic surfaces holomorphically dominable by* $\mathbf{C}^2$, Invent. Math. **139** (2000), no. 3, 617–659, https://doi.org/10.1007/s002220050021. ↑ 27

[20] F. Forstnerič, *Stein Manifolds and Holomorphic Mappings: The Homotopy Principle in Complex Analysis*, 2nd ed., Springer, Cham, 2017. ↑ 2, 22, 34, 64, 67

[21] Y. Kusakabe, *Elliptic characterization and localization of Oka manifolds*, Indiana Univ. Math. J. **70** (2021), no. 3, 1039–1054, https://doi.org/10.1512/IUMJ.2021.70.8454. ↑ 2, 67

[22] Z. Chen, *A Conditional Oka Complex Structure of the Six-Sphere*, arXiv:2609.26706v1, submitted 22 September 2026, https://arxiv.org/abs/2609.26706. ↑ 105

[23] F. Forstnerič, *Oka manifolds: From Oka to Stein and back*, Annales de la Faculté des Sciences de Toulouse, Mathématiques **22** (2013), no. 4, 747–809; Theorem 4.1 and p. 803, https://www.numdam.org/item/AFST_2013_6_22_4_747_0/. ↑ 2, 35, 96

[24] Y. Kusakabe, *Elliptic characterization and unification of Oka maps*, Math. Z. **298** (2021), 1735–1750, https://doi.org/10.1007/s00209-020-02661-y. Theorem numbering follows https://arxiv.org/abs/2003.03757v1. ↑ 2, 22, 34

[25] O. Forster, *Lectures on Riemann Surfaces*, Graduate Texts in Mathematics, vol. 81, Springer, New York, 1981. ↑ 35

[26] J.-P. Rosay and W. Rudin, *Holomorphic maps from* $\mathbf{C}^n$ *to* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **310** (1988), no. 1, 47–86, https://doi.org/10.1090/S0002-9947-1988-0929658-4. ↑ 35

[27] F. Lárusson and T. T. Truong, *Algebraic subellipticity and dominability of blow-ups of affine spaces*, Doc. Math. **22** (2017), 151–163, https://doi.org/10.4171/DM/562. ↑ 67, 73

[28] F. Forstnerič and F. Lárusson, *Oka–1 manifolds: New examples and properties*, Math. Z. **309** (2025), article 26, https://doi.org/10.1007/s00209-024-03649-8. ↑ 73

[29] S. Kaliman and M. Zaidenberg, *Gromov ellipticity and subellipticity*, Forum Math. **36** (2024), no. 2, 373–376, https://doi.org/10.1515/forum-2023-0016. ↑ 73

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